Journal of
Mechanics of
Materials and Structures

ROBUSTNESS ANALYSIS OF STRUCTURES BASED ON PLASTIC LIMIT ANALYSIS WITH UNCERTAIN LOADS

Yu Matsuda and Yoshihiro Kanno

Volume 3, № 2 February 2008

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 3, No. 2, 2008

ROBUSTNESS ANALYSIS OF STRUCTURES BASED ON PLASTIC LIMIT ANALYSIS WITH UNCERTAIN LOADS

YU MATSUDA AND YOSHIHIRO KANNO

This paper presents a method for computing an info-gap robustness function of structures, which is regarded as one measure of structural robustness, under uncertainties associated with the limit load factor. We assume that the external load in the plastic limit analysis is uncertain around its nominal value. Various uncertainties are considered for the live, dead, and reference disturbance loads based on the nonstochastic info-gap uncertainty model. Although the robustness function is originally defined by using the optimization problem with infinitely many constraints, we show that the robustness function is obtained as an optimal value of a linear programming (LP) problem. Hence, we can easily compute the info-gap robustness function associated with the limit load factor by solving an LP problem. As the second contribution, we show that the robust structural optimization problems of trusses and frames can also be reduced to LP problems. In numerical examples, the robustness functions, as well as the robust optimal designs, are computed for trusses and framed structures by solving LP problems.

1. Introduction

In designing civil, mechanical and aerospace structures, plastic limit analysis has been used widely for decades as a means of estimating the ultimate strength of structures. On the other hand, structural analysis considering the uncertainties have received fast-growing interest, because structures that are actually built will always have various uncertainties caused by manufacturing errors, limitation of knowledge of input disturbances, observation errors, simplification for modeling, etc. This paper discusses a solution technique for computing the measure of robustness of structures, where the applied loads are supposed to be uncertain. We assume that the dead, live, and/or the reference disturbance loads in limit analysis are uncertain around their nominal values. It should be emphasized that arbitrarily large uncertainty of these loads can be dealt with in our framework.

The limit analysis still receives much attention by numerous researchers from the viewpoint of solution techniques [Muralidhar and Jagannatha Rao 1997; Andersen et al. 1998; Cocchetti and Maier 2003; Krabbenhoft and Damkilde 2003]. Based on the probabilistic uncertainty models of structural systems, various approaches to stochastic limit analysis have also been proposed [Llyoyd Smith et al. 1990; Rocho and Sonnenberg 2003; Staat and Heitzer 2003; Marti and Stoeckel 2004]. Recently, based on the non-probabilistic uncertainty model, Kanno and Takewaki [2007] has proposed a global optimization method for computing the smallest limit load factor of truss structures, in which the applied dead load is assumed to be uncertain but bounded.

Reliability-based structural design methods have been investigated extensively based on the framework of probabilistic uncertainty models [Kharmanda et al. 2004; Zang et al. 2005]. Nonprobabilistic

Keywords: data uncertainty, linear program, plastic limit analysis, robust optimization, info-gap analysis.

uncertainty models have also been developed for uncertain structural analysis. In such a nonprobabilistic uncertainty model, a mechanical system contains some unknown parameters which are assumed to be bounded. Ben-Haim and Elishakoff [1990] developed the well-known *convex model* approach, which has been applied to a robust truss optimization by Ganzerli and Pantelides [1999]. The interval linear algebra has been well developed for the so-called uncertain linear equations [Alefeld and Mayer 2000], which were employed in structural analysis considering various uncertainties [Qiu and Elishakoff 1998; Muhanna and Mullen 2001; Chen et al. 2002].

Recently, the *info-gap decision theory* has been proposed as a nonprobabilistic decision theory under uncertainties [Ben-Haim 2006], and has been applied to wide fields. In the info-gap decision theory, the *robustness function* plays a key role as a measure of robustness of systems having uncertainties [Ben-Haim 2006]. In structural engineering, the info-gap robustness function represents the greatest level of uncertainty at which any constraint on mechanical performance cannot be violated. The constraints on mechanical performance can be violated only at the large level of uncertainty in a structure with a large robustness function, while they can possibly be violated at a small level of uncertainty in a structure with a small robustness function. Thus, we can compare robustness of structures quantitatively in terms of the robustness function.

Unfortunately, in many practical situations it is difficult to compute the exact value of the robustness function of a structure. This is because the robustness function is defined as the optimal value of an optimization problem with infinitely many constraints. Kanno and Takewaki [2006a] proposed a method for computing a lower bound of the robustness function for trusses associated with stress and/or displacement constraints. Takewaki and Ben-Haim [2005] computed the robustness function of damped structures considering the dynamic response constraints. In the case of Takewaki and Ben-Haim [2005], the *worst case* of the uncertain parameters can be obtained analytically, which enables us to compute the exact value of the robustness function.

In this paper, we investigate the info-gap robustness function of structures associated with the lower bound constraint on the limit load factor. In the plastic limit analysis, we consider the uncertainties of the dead, live, and/or the reference disturbance loads, which obey the info-gap uncertainty models. As a main contribution, we show that the robustness function considering the limit load factor constraint can be obtained as an optimal value of a *linear programming* (LP) problem, which implies that the exact value of the robustness function can be computed easily. This is rather amazing, because it is not straightforward to find the worst case of the limit load factor under the uncertainty of dead load. Indeed, we have to find the global optimal solution of a nonlinear optimization problem in order to detect the worst-case limit load factor [Kanno and Takewaki 2007]. Thus, the results of this paper imply that computing the robustness function is much easier than finding the worst-case limit load factor. Consequently, there exists a class of constraints such that the robustness function can be computed easily while it is very difficult to find the worst case.

As the second contribution, we formulate the *robust counterpart* to the structural optimization associated with the limit load factor and present its tractable reformulation. For convex optimization problems, the notion and methodology of robust counterpart problem were developed by Ben-Tal and Nemirovski [2002], and were applied to robust compliance minimization of trusses [Ben-Tal and Nemirovski 1997]. As an alternative approach, robust optimization problems were formulated for structures based on the convex model analysis [Elishakoff et al. 1994; Ganzerli and Pantelides 1999], provided that the variations of uncertain parameters are sufficiently small. The maximization problem of the robustness function of trusses associated with stress constraints was studied in [Kanno and Takewaki 2006b]. For a comprehensive survey on the robust structural design, the readers may refer to the review papers [Zang et al. 2005; Beyer and Sendhoff 2007].

For the limit load factor constraint, we formulate the minimization problem of the structural volume under the constraint such that the lower bound constraint on the limit load factor is always satisfied for the given level of uncertainty. In this problem, the major difficulty arises where the constraint includes the sublevel optimization problem even in the nominal case, because the limit load factor is defined as an optimal value of an optimization problem. It is shown that this robust optimization problem can be reformulated as an LP problem for trusses as well as frames with sandwich cross-sections.

This paper is organized as follows. In Section 2 we prepare the LP problem for the conventional limit analysis and introduce the definition of robustness function as well as the info-gap uncertainty model for structural analysis. For trusses, the robustness function associated with the lower bound constraint on the limit load factor is defined in Section 3 for various uncertainty models of external load, and for each model an LP problem is formulated which provides the robustness function. In Section 4, we show that the robustness function for a framed structure can be computed by solving an LP problem. Numerical experiments are presented in Section 5 for a truss and frames. The robust optimization problems associated with the limit load factor are formulated for trusses and frames in Sections 6.1 and 6.2, respectively, and they are reformulated into LP problems. Numerical experiments are presented in Section 6.3 for robust structural optimization, while conclusions are drawn in Section 7.

2. Preliminaries

2.1. ***Notation.*** The ℓ^p-norm of the vector $\boldsymbol{x} = (x_i) \in \Re^n$ for $1 \leq p < \infty$ is defined as

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

In particular, the ℓ^1- and ℓ^2-norms are written as

$$\|\boldsymbol{p}\|_1 = \sum_{i=1}^{n} |p_i|, \quad \|\boldsymbol{p}\|_2 = (\boldsymbol{p}^\top \boldsymbol{p})^{1/2}.$$

The ℓ^∞-norm is defined as $\|\boldsymbol{p}\|_\infty = \max_{i \in \{1,\dots,n\}} |p_i|$. For p satisfying $1 < p < \infty$, p^* is defined by

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

For $p = 1$ and $p = \infty$, we simply set $p^* = \infty$ and $p^* = 1$, respectively.

For column vectors $\boldsymbol{p} = (p_i) \in \Re^m$ and $\boldsymbol{q} = (q_i) \in \Re^n$, the $(m+n)$-dimensional column vector $(\boldsymbol{p}^\top, \boldsymbol{q}^\top)^\top$ is often written simply as $(\boldsymbol{p}, \boldsymbol{q})$. We write $\boldsymbol{p} \geq \boldsymbol{0}$ if $p_i \geq 0$ $(i = 1, \dots, m)$. Define $\Re^n_+$ and $\Re^n_{++}$ by

$$\Re^n_+ = \{\boldsymbol{x} \in \Re^n \,|\, \boldsymbol{x} \geq \boldsymbol{0}\},$$
$$\Re^n_{++} = \{\boldsymbol{x} = (x_i) \in \Re^n \,|\, x_i > 0 \quad (i = 1, \dots, n)\}.$$

The two sets $\mathcal{A} \subseteq \Re^m$ and $\mathcal{B} \subseteq \Re^n$ have Cartesian products defined by

$$\mathcal{A} \times \mathcal{B} = \{(\boldsymbol{a}^\top, \boldsymbol{b}^\top)^\top \in \Re^{m+n} \,|\, \boldsymbol{a} \in \mathcal{A},\ \boldsymbol{b} \in \mathcal{B}\}.$$

In particular, we write $\Re^{m+n} = \Re^m \times \Re^n$. The empty set is denoted by $\varnothing$.

2.2. ***Robustness function associated with limit load factor.*** The robustness function was proposed as a measure of robustness for a general uncertain system, whose uncertainty is described by an info-gap uncertainty model [Ben-Haim 2006]. In this section, we formulate the robustness function of engineering structures for a particular case in which the limit load factor is chosen as a measure of structural performance.

Consider a finitely discretized structure. Small rotations and small strains are assumed. Let $\boldsymbol{f} \in \Re^{n^{\mathrm{d}}}$ denote the vector of the external forces, where n^{d} denotes the number of degrees of freedom of displacements. Suppose that $\boldsymbol{f}$ consists of the constant part $\boldsymbol{f}_{\mathrm{D}}$ and proportionally increasing part $\lambda \boldsymbol{f}_{\mathrm{R}}$ as

$$\boldsymbol{f} = \lambda \boldsymbol{f}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}. \tag{1}$$

Notice here that $\lambda \boldsymbol{f}_{\mathrm{R}}$ is defined by the monotonically increasing load parameter $\lambda \in \Re_+$ and the constant reference load $\boldsymbol{f}_{\mathrm{R}} \in \Re^{n^{\mathrm{d}}} \setminus \{\boldsymbol{0}\}$. In civil engineering, $\boldsymbol{f}_{\mathrm{D}}$ consists of the dead load, live load, etc., while $\lambda \boldsymbol{f}_{\mathrm{R}}$ is referred to as the live or disturbance load which may be a static approximation of dynamical loads caused by earthquakes, winds, etc. In this paper, $\boldsymbol{f}_{\mathrm{D}}$ is simply called the *dead load* and $\boldsymbol{f}_{\mathrm{R}}$ is called the *reference disturbance load* for simplicity of presentation.

For the given $\boldsymbol{f}_{\mathrm{R}}$ and $\boldsymbol{f}_{\mathrm{D}}$, let $\lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}})$ denote the limit load factor. Throughout the paper, we assume $\lambda^*(\boldsymbol{0}, \boldsymbol{f}_{\mathrm{D}}) > 0$, that is, the plastic collapse does not occur with the dead load $\boldsymbol{f}_{\mathrm{D}}$ only. Let $\underline{\lambda}$ denote the lower bound of the limit load factor, which is the performance requirement imposed by engineers. For the given $\underline{\lambda} \in \Re_{++}$, the conventional constraint on the limit load factor is written as

$$\lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}}) \geq \underline{\lambda}. \tag{2}$$

We next suppose that $\boldsymbol{f}_{\mathrm{R}}$ and $\boldsymbol{f}_{\mathrm{D}}$ are known imprecisely. Let $\tilde{\boldsymbol{f}}_{\mathrm{R}} \in \Re^{n^{\mathrm{d}}}$ and $\tilde{\boldsymbol{f}}_{\mathrm{D}} \in \Re^{n^{\mathrm{d}}}$ denote the nominal values (or the best estimates) of $\boldsymbol{f}_{\mathrm{R}}$ and $\boldsymbol{f}_{\mathrm{D}}$, respectively. For the given $\alpha \in \Re_+$, $\tilde{\boldsymbol{f}}_{\mathrm{R}}$, and $\tilde{\boldsymbol{f}}_{\mathrm{D}}$, let $\mathcal{R}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{R}}) \subset \Re^{n^{\mathrm{d}}}$ and $\mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{D}}) \subset \Re^{n^{\mathrm{d}}}$ be bounded sets. The rigorous and concrete definitions of $\mathcal{R}_p$ and $\mathcal{D}_p$ will be given in Section 3. The subscript p of $\mathcal{R}_p$ and $\mathcal{D}_p$ implies that the sets $\mathcal{R}_p$ and $\mathcal{D}_p$ are defined by using the ℓ^p-norm $(1 \leq p \leq +\infty)$ as shown below. The parameter α represents the magnitude of the uncertainty, and hence α is referred to as the *uncertainty parameter* [Ben-Haim 2006].

The uncertainties of $\boldsymbol{f}_{\mathrm{R}}$ and $\boldsymbol{f}_{\mathrm{D}}$ are modeled as follows. For any $\boldsymbol{f}_{\mathrm{R}}$ and $\boldsymbol{f}_{\mathrm{D}}$, assume that there exists an $\alpha \in \Re_+$ such that the conditions

$$\boldsymbol{f}_{\mathrm{R}} \in \mathcal{R}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{R}}), \quad \boldsymbol{f}_{\mathrm{D}} \in \mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{R}}), \tag{3}$$

are satisfied. We call $\mathcal{R}_p$ and $\mathcal{D}_p$ the uncertainty sets of $\boldsymbol{f}_{\mathrm{R}}$ and $\boldsymbol{f}_{\mathrm{D}}$, respectively. We further assume that $\mathcal{R}_p$ and $\mathcal{D}_p$ satisfy the two basic axioms of the info-gap model [Ben-Haim 2006]:

(i) Nesting: $0 \leq \alpha_1 < \alpha_2$ implies $\mathcal{R}_p(\alpha_1, \tilde{\boldsymbol{f}}_{\mathrm{R}}) \times \mathcal{D}_p(\alpha_1, \tilde{\boldsymbol{f}}_{\mathrm{D}}) \subset \mathcal{R}_p(\alpha_2, \tilde{\boldsymbol{f}}_{\mathrm{R}}) \times \mathcal{D}_p(\alpha_2, \tilde{\boldsymbol{f}}_{\mathrm{D}})$,

(ii) Contraction: $\mathcal{R}_p(0, \tilde{\boldsymbol{f}}_{\mathrm{R}}) = \{\tilde{\boldsymbol{f}}_{\mathrm{R}}\}$ and $\mathcal{D}_p(0, \tilde{\boldsymbol{f}}_{\mathrm{D}}) = \{\tilde{\boldsymbol{f}}_{\mathrm{D}}\}$.

From the nesting axiom we see that the uncertainty sets $\mathcal{R}_p(\alpha, \tilde{\boldsymbol{f}}_\mathrm{R})$ and $\mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_\mathrm{D})$ become more inclusive as α becomes larger. The greater the value of α, the greater the ranges of possible variations of $\boldsymbol{f}_\mathrm{R}$ and $\boldsymbol{f}_\mathrm{D}$. The contraction axiom guarantees that the estimates $\tilde{\boldsymbol{f}}_\mathrm{R}$ and $\tilde{\boldsymbol{f}}_\mathrm{D}$ are exact at $\alpha = 0$. Note that the value of α is usually unknown in actual structures. Throughout the following robustness analysis we do not use any knowledge of the actual range of uncertainty of loads, which is regarded as one of the advantages of using the info-gap theory.

For the fixed $\alpha \in \Re_+$, the robust counterpart of the constraint (2), is written as

$$\lambda^*(\boldsymbol{f}_\mathrm{R}, \boldsymbol{f}_\mathrm{D}) \geq \underline{\lambda}, \quad \text{for all } \boldsymbol{f}_\mathrm{R} \in \mathcal{R}_p(\alpha, \tilde{\boldsymbol{f}}_\mathrm{R}), \quad \text{for all } \boldsymbol{f}_\mathrm{D} \in \mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_\mathrm{D}). \tag{4}$$

Throughout the paper, we assume $\lambda^*(\tilde{\boldsymbol{f}}_\mathrm{R}, \tilde{\boldsymbol{f}}_\mathrm{D}) \geq \underline{\lambda}$, that is, the robust constraint, Equation (4), is satisfied at the nominal situation. The robustness function represents the largest value of α with which the robust constraint, (4), is satisfied. More precisely, the robustness function $\hat{\alpha} : \Re_+ \to [0, +\infty]$ associated with the constraint of the limit load factor is defined as

$$\hat{\alpha}(\underline{\lambda}) = \max \left\{ \alpha \;\middle|\; \lambda^*(\boldsymbol{f}_\mathrm{R}, \boldsymbol{f}_\mathrm{D}) \geq \underline{\lambda} \quad \left(\text{for all } (\boldsymbol{f}_\mathrm{R}, \boldsymbol{f}_\mathrm{D}) \in \mathcal{R}_p(\alpha, \tilde{\boldsymbol{f}}_\mathrm{R}) \times \mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_\mathrm{D}) \right) \right\}. \tag{5}$$

Thus, $\hat{\alpha}$ is the function of the performance requirement $\underline{\lambda}$, as well as of the design variables of the structure. For the fixed $\underline{\lambda}$, the constraint, Equation (4), can be violated only at a large level of uncertainty if the structure has a large value of $\hat{\alpha}(\underline{\lambda})$. On the other hand, (4) can be violated at a small level of uncertainty if the structure has a small value of $\hat{\alpha}(\underline{\lambda})$. In this way, we can compare robustness of structures quantitatively in terms of the robustness function.

The problem (5), is classified to the semiinfinite programming, which means an optimization problem having a finite number of variables and infinitely many inequality constraints. Unfortunately, it is difficult to solve (5) directly, which motivates us to investigate a tractable reformulation in the following sections.

3. Robustness analysis of trusses

We investigate a tractable reformulation of the info-gap robustness function of trusses under various uncertainty models of external loads.

3.1. ***Basic problem of limit analysis of trusses.*** In this section, in order to make the paper self-contained, we prepare an LP problem for the conventional limit analysis of trusses. Consider an elastic/perfectly-plastic truss in the two- or three-dimensional space. Let n^m denote the number of members. We denote by $\boldsymbol{q} = (q_i) \in \Re^{n^\mathrm{m}}$ the vector of member axial forces. The system of equilibrium equations between $\boldsymbol{q}$ and the external load $\boldsymbol{f}$ are written in the form of

$$H\boldsymbol{q} = \boldsymbol{f}, \tag{6}$$

where $H \in \Re^{n^\mathrm{d} \times n^\mathrm{m}}$ is a constant matrix. Recall that $\boldsymbol{f}$ is divided into two parts as Equation (1).

Let $\bar{\sigma}_i > 0$ and $-\bar{\sigma}_i$ denote the yield stresses of the ith member in tension and compression, respectively. Here, we assume for simplicity that the yield stresses in tension and compression share a common absolute value. The member cross-sectional area is denoted by $a_i > 0$. Define $\bar{q}_i$ by

$$\bar{q}_i = \bar{\sigma}_i a_i, \tag{7}$$

which is the absolute value of the admissible axial force. The admissible set $\mathcal{Q} \subset \Re^{n^{\mathrm{m}}}$ of the axial forces is written as

$$\mathcal{Q} = \left\{ \boldsymbol{q} \in \Re^{n^{\mathrm{m}}} \,\middle|\, \bar{q}_i \geq |q_i| \quad (i = 1, \dots, n^{\mathrm{m}}) \right\}. \tag{8}$$

From the static (or lower bound) principle [Hodge 1959], and by using Equation (1), (6), and (8), the limit load factor λ^* is obtained by solving the following LP problem:

$$\lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}}) = \max_{\lambda, \boldsymbol{q}} \left\{ \lambda \,\middle|\, H\boldsymbol{q} = \lambda \boldsymbol{f}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}, \quad \boldsymbol{q} \in \mathcal{Q} \right\}, \tag{9}$$

where the variables are λ and $\boldsymbol{q}$. Note that the limit load factor λ^* is regarded as a function of $\boldsymbol{f}_{\mathrm{R}}$ and $\boldsymbol{f}_{\mathrm{D}}$. Then the robustness function of the truss is defined as Equation (5). In the following discussion, we consider various models of the uncertainty sets $\mathcal{R}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{R}})$ and $\mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{D}})$, and discuss how to compute the robustness function $\hat{\alpha}(\underline{\lambda})$.

3.2. *Uncertainty of dead load.* In this section, we suppose that the dead load $\boldsymbol{f}_{\mathrm{D}}$ possesses uncertainty, while the reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$ is assumed to be certain. Let $\boldsymbol{\zeta} \in \Re^{n^{\mathrm{z}}}$ denote the vector of parameters that are considered to be unknown, or uncertain, where n^{z} denotes the number of such parameters. We describe the uncertainty of $\boldsymbol{f}_{\mathrm{D}}$ in terms of the unknown $\boldsymbol{\zeta}$. Suppose that $\boldsymbol{f}_{\mathrm{D}}$ depend on $\boldsymbol{\zeta}$ affinely so that the uncertainty set in Equation (3) is defined as

$$\mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{D}}) = \left\{ \boldsymbol{f}_{\mathrm{D}} \in \Re^{n^{\mathrm{d}}} \,\middle|\, \boldsymbol{f}_{\mathrm{D}} = \tilde{\boldsymbol{f}}_{\mathrm{D}} + F_0 \boldsymbol{\zeta}, \quad \alpha \geq \|\boldsymbol{\zeta}\|_p \right\}, \tag{10}$$

where $1 \leq p \leq +\infty$. Note that Equation (10) is the unified description of uncertainty models defined by using various norms, that is, the choice of p provides us with a variety of uncertainty models. In the uncertainty set, Equation (10), the constant matrix $F_0 \in \Re^{n^{\mathrm{d}} \times n^{\mathrm{z}}}$ represents the relative magnitude of the uncertainty of $f_{\mathrm{D}j}$ and the correlation of the uncertainties among $f_{\mathrm{D}1}, \dots, f_{\mathrm{D}n^{\mathrm{d}}}$. Each component of F_0 has the unit of force. Hence, neither $\boldsymbol{\zeta}$ nor α has no physical unit. It is easy to verify that the uncertainty set $\mathcal{D}_p$ defined by Equation (10) satisfies the axioms of the info-gap model introduced in Section 2.2.

An example of a truss is illustrated in Figure 1. To impose a nominal dead load $\tilde{\boldsymbol{f}}_{\mathrm{D}}$, we suppose that external forces are applied at the nodes (f) and (g). The nominal reference disturbance load $\tilde{\boldsymbol{f}}_{\mathrm{R}}$ is defined such that the proportionally increasing forces are applied at the nodes (c) and (d). In order to guarantee that $\boldsymbol{f}_{\mathrm{R}}$ is certain, F_0 is assumed to satisfy the condition that the components of $F_0\,\boldsymbol{\zeta}$ corresponding to the external forces applied to the nodes (c) and (d) vanish for any $\boldsymbol{\zeta} \in \Re^{n^{\mathrm{z}}}$.

According to Equation (5), the robustness function $\hat{\alpha}{:}\Re_+ \to [0, +\infty]$ with the uncertainty model $\boldsymbol{f}_{\mathrm{D}} \in \mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{D}})$ is defined as

$$\hat{\alpha}(\underline{\lambda}) = \max_{\alpha} \left\{ \alpha \,\middle|\, \lambda^*(\tilde{\boldsymbol{f}}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}}) \geq \underline{\lambda} \quad \left(\text{for all } \boldsymbol{f}_{\mathrm{D}} \in \mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{D}}) \right) \right\}. \tag{11}$$

For simplicity, we write $\boldsymbol{f}_{\mathrm{R}} = \tilde{\boldsymbol{f}}_{\mathrm{R}}$ in the remainder of this section.

For $r \in \Re_+$, we define the set $\mathcal{B}_p(r) \subset \Re^{n^{\mathrm{z}}}$ by

$$\mathcal{B}_p(r) = \left\{ \boldsymbol{\zeta} \in \Re^{n^{\mathrm{z}}} \,\middle|\, r \geq \|\boldsymbol{\zeta}\|_p \right\}.$$

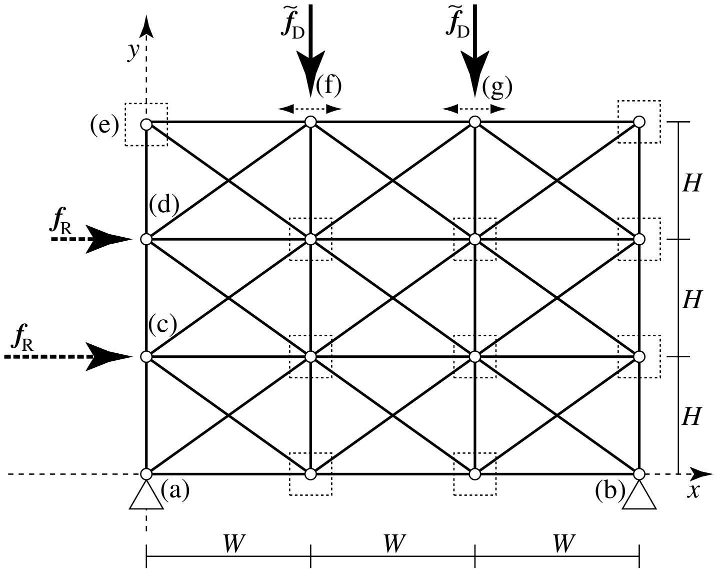

Figure 1. 3×3 truss.

The following proposition prepares the reformulation of the semiinfinite optimization problem (11) by eliminating λ^*.

Proposition 3.1. Define $\hat{r}$ by

$$\hat{r} = \max_{r,\boldsymbol{q}} \left\{ r \;\middle|\; H\boldsymbol{q} = \underline{\lambda}\boldsymbol{f}_{\rm R} + \tilde{\boldsymbol{f}}_{\rm D} + F_0\boldsymbol{\zeta}, \quad \boldsymbol{q} \in \mathcal{Q} \quad (\text{for all } \boldsymbol{\zeta} \in \mathcal{B}_p(r)) \right\}. \tag{12}$$

Then the robustness function $\hat{\alpha}(\underline{\lambda})$ defined by Equation (10) and (11) satisfies $\hat{\alpha}(\underline{\lambda}) = \hat{r}$.

Proof. The constraint of (12) implies that the condition

$$\exists\, \boldsymbol{q}' \in \Re^{n^{\rm m}}: \quad H\boldsymbol{q}' = \underline{\lambda}\boldsymbol{f}_{\rm R} + \tilde{\boldsymbol{f}}_{\rm D} + F_0\boldsymbol{\zeta}, \quad \boldsymbol{q}' \in Q \tag{13}$$

is satisfied for any $\boldsymbol{\zeta}$ satisfying $\hat{r} > \|\boldsymbol{\zeta}\|_p$. For a fixed $\lambda \in \Re_+$, define the set $\mathcal{V}(\lambda) \subset \Re^{n^{\rm m}}$ as

$$\mathcal{V}(\lambda) = \left\{ \boldsymbol{q} \in \Re^{n^{\rm m}} \;\middle|\; H\boldsymbol{q} = \lambda\boldsymbol{f}_{\rm R} + \tilde{\boldsymbol{F}}_{\rm D}, \quad \boldsymbol{q} \in Q \right\}.$$

Note that $\boldsymbol{q} \in \mathcal{V}(\lambda)$ if and only if $(\lambda, \boldsymbol{q})$ is a feasible solution of the problem, Equation (9). Since (13) holds for $\boldsymbol{\zeta} = \boldsymbol{0}$, we see that $\boldsymbol{q}' \in \mathcal{V}(\underline{\lambda})$ is satisfied. From this observation and (9), we obtain

$$\lambda^*(\boldsymbol{f}_{\rm R}, \boldsymbol{f}_{\rm D}(\boldsymbol{\zeta})) \geq \underline{\lambda}, \tag{14}$$

where $\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta}) = \tilde{\boldsymbol{f}}_{\rm D} + F_0\boldsymbol{\zeta}$. Since we can show that Equation (14) holds for any $\boldsymbol{\zeta}$ satisfying $\hat{r} > \|\boldsymbol{\zeta}\|_p$, the definition (11) of $\hat{\alpha}$ implies

$$\hat{\alpha}(\underline{\lambda}) \geq \hat{r}. \tag{15}$$

On the other hand, choose $\boldsymbol{\zeta}'$ satisfying $\hat{\alpha} > \boldsymbol{\zeta}'$. It follows from Equation (11) that $\mathcal{V}(\lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}'))) \neq \varnothing$ and $\mathcal{V}(0) \neq \varnothing$. Moreover, the set $\{(\lambda, \boldsymbol{q}) \in \Re \times \Re^{n^{\mathrm{m}}} \mid \boldsymbol{q} \in \mathcal{V}(\lambda)\}$ is convex, from which it follows that for any $\underline{\lambda}$ satisfying $0 \leq \underline{\lambda} \leq \lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}}(\boldsymbol{\zeta}'))$, $\mathcal{V}(\underline{\lambda}) \neq \varnothing$ is satisfied, that is, Equation (13) is satisfied. Since this observation holds for any $\boldsymbol{\zeta}'$ satisfying $\hat{\alpha} > \|\boldsymbol{\zeta}'\|_p$, the definition (12) of $\hat{r}$ implies

$$\hat{\alpha}(\underline{\lambda}) \leq \hat{r}. \tag{16}$$

Consequently, from Equation (15) and (16) we obtain $\hat{\alpha}(\underline{\lambda}) = \hat{r}$, which concludes the proof. □

It is still difficult to solve Equation (12) because it requires that the constraints hold for infinitely many $\boldsymbol{\zeta}$ satisfying $\boldsymbol{\zeta} \in \mathcal{B}_p(r)$.

Let $H^\dagger \in \Re^{n^{\mathrm{m}} \times n^{\mathrm{d}}}$ denote the pseudoinverse of H. A basis for the null space of H is denoted by $H^\perp \in \Re^{n^{\mathrm{m}} \times n^\xi}$, where $n^\xi = n^{\mathrm{m}} - \mathrm{rank}(H)$. Let $\boldsymbol{h}_i^\dagger$ and $\boldsymbol{h}_i^\perp$ the ith row vectors of $H^\dagger$ and $H^\perp$, respectively, that is,

$$H^\dagger = \begin{pmatrix} \boldsymbol{h}_1^\dagger \\ \vdots \\ \boldsymbol{h}_{n^{\mathrm{m}}}^\dagger \end{pmatrix}, \quad H^\perp = \begin{pmatrix} \boldsymbol{h}_1^\perp \\ \vdots \\ \boldsymbol{h}_{n^{\mathrm{m}}}^\perp \end{pmatrix}.$$

Proposition 3.2. $\hat{r}$ defined by Equation (12) is equal to the optimal value of the LP problem

$$\begin{aligned} \hat{r} = \max_{r,\boldsymbol{\xi}} \Big\{ r \,\Big|\, & \boldsymbol{h}_i^\dagger(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}} + \tilde{\boldsymbol{f}}_{\mathrm{D}}) + r\|\boldsymbol{h}_i^\dagger F_0\|_{p^*} + \boldsymbol{h}_i^\perp \boldsymbol{\xi} \leq \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}}, \\ & -\boldsymbol{h}_i^\dagger(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}} + \tilde{\boldsymbol{f}}_{\mathrm{D}}) + r\|\boldsymbol{h}_i^\dagger F_0\|_{p^*} - \boldsymbol{h}_i^\perp \boldsymbol{\xi} \leq \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}} \Big\} \end{aligned} \tag{17}$$

in the variables $r \in \Re$ and $\boldsymbol{\xi} \in \Re^{n^\xi}$.

Proof. Observe that any $\boldsymbol{q} \in \Re^{n^m}$ satisfying the equilibrium equations

$$H\boldsymbol{q} = \underline{\lambda}\boldsymbol{f}_{\mathrm{R}} + \tilde{\boldsymbol{f}}_{\mathrm{D}} + F_0\boldsymbol{\zeta},$$

can be represented as

$$\boldsymbol{q} = H^\dagger(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}} + \tilde{\boldsymbol{f}}_{\mathrm{D}} + F_0\boldsymbol{\zeta}) + H^\perp\boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \Re^{n^\xi}. \tag{18}$$

In Equation (18), we may regard q_i as a function of $\boldsymbol{\zeta}$, that is,

$$q_i(\boldsymbol{\zeta}) := \boldsymbol{h}_i^\dagger(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}} + \tilde{\boldsymbol{f}}_{\mathrm{D}}) + \boldsymbol{h}_i^\dagger F_0\boldsymbol{\zeta} + \boldsymbol{h}_i^\perp\boldsymbol{\xi}, \quad i = 1, \dots, n^{\mathrm{m}}. \tag{19}$$

From the definition Equation (8) of $\mathcal{Q}$ it follows that the constraints of (12) are equivalently rewritten as

$$q_i(\boldsymbol{\zeta}) \leq \bar{q}_i \quad (\text{for all } \boldsymbol{\zeta} \in \mathcal{B}_p(r)), \quad i = 1, \dots, n^{\mathrm{m}}, \tag{20}$$

$$-q_i(\boldsymbol{\zeta}) \leq \bar{q}_i \quad (\text{for all } \boldsymbol{\zeta} \in \mathcal{B}_p(r)), \quad i = 1, \dots, n^{\mathrm{m}}. \tag{21}$$

Moreover, Equation (20) and (21) are equivalent to

$$\begin{aligned} \max_{\boldsymbol{\zeta}} \left\{ q_i(\boldsymbol{\zeta}) \,\middle|\, r \geq \|\boldsymbol{\zeta}\|_p \right\} \leq \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}}, \\ \max_{\boldsymbol{\zeta}} \left\{ -q_i(\boldsymbol{\zeta}) \,\middle|\, r \geq \|\boldsymbol{\zeta}\|_p \right\} \leq \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}}. \end{aligned} \tag{22}$$

By using Equation (19) and the Hölder inequality [Michael Sttele 2004], we see that

$$\max_{\zeta}\left\{\boldsymbol{h}_i^{\dagger}F_0\boldsymbol{\zeta}\,\middle|\, r\geq\|\boldsymbol{\zeta}\|_p\right\}=r\|\boldsymbol{h}_i^{\dagger}F_0\|_{p^*},\tag{23}$$

from which it follows that (22) is equivalently rewritten as

$$\begin{aligned}\boldsymbol{h}_i^{\dagger}(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}}+\tilde{\boldsymbol{f}}_{\mathrm{D}})+r\|\boldsymbol{h}_i^{\dagger}F_0\|_{p^*}+\boldsymbol{h}_i^{\perp}\boldsymbol{\xi}\leq\bar{q}_i,\quad i=1,\dots,n^{\mathrm{m}},\\-\boldsymbol{h}_i^{\dagger}(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}}+\tilde{\boldsymbol{f}}_{\mathrm{D}})+r\|\boldsymbol{h}_i^{\dagger}F_0\|_{p^*}-\boldsymbol{h}_i^{\perp}\boldsymbol{\xi}\leq\bar{q}_i,\quad i=1,\dots,n^{\mathrm{m}}.\end{aligned}\tag{24}$$

Consequently, the constraints of Equation (12) are equivalent to (24), which concludes the proof. □

Proposition 3.2, together with Proposition 3.1, implies that the robustness function $\hat{\alpha}(\underline{\lambda})$ can be obtained easily by solving an LP problem (17), contradicting the fact that it is very difficult to solve the semiinfinite optimization problem (11).

3.3. *Uncertainty of reference disturbance load.* In this section, we investigate the uncertainty model of the reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$ in Equation (1), while the dead load $\boldsymbol{f}_{\mathrm{D}}$ is assumed to be certain. For the given nominal value $\tilde{\boldsymbol{f}}_{\mathrm{R}}$ and fixed $\alpha\in\Re_+$, let $\mathcal{R}_p(\alpha,\tilde{\boldsymbol{f}}_{\mathrm{R}})$ denote the uncertainty set of $\boldsymbol{f}_{\mathrm{R}}$, which shall be rigorously defined below. According to Equation (5), the robustness function $\hat{\alpha}:\Re_+\to[0,+\infty]$ in this case is defined as

$$\hat{\alpha}(\underline{\lambda})=\max_{\alpha}\left\{\alpha\,\middle|\,\lambda^*(\boldsymbol{f}_{\mathrm{R}},\tilde{\boldsymbol{f}}_{\mathrm{D}})\geq\underline{\lambda}\ \left(\text{for all } \boldsymbol{f}_{\mathrm{R}}\in\mathcal{R}_p(\alpha,\tilde{\boldsymbol{f}}_{\mathrm{R}})\right)\right\}.\tag{25}$$

For simplicity, we write $\boldsymbol{f}_{\mathrm{D}}=\tilde{\boldsymbol{f}}_{\mathrm{D}}$ in the remainder of this section .

3.3.1. *Uncertainty of load distribution.* Consider the uncertainty (or variations) of the distribution of the reference disturbance load.

Recall the example of a truss illustrated in Figure 1, which has been studied for the uncertainty model $\mathcal{D}_p$ in Equation (10). At the nodes of the left side, the external forces are applied as the reference disturbance load. The nominal forces applied at the nodes (c) and (d) are illustrated in Figure 2 as $f_{\mathrm{R}}^{(\mathrm{c})}$ and $f_{\mathrm{R}}^{(\mathrm{d})}$. Suppose that the directions of these forces do not change, while the distribution is unknown as shown in Figure 2 as $f_{\mathrm{R}}^{(\mathrm{c})}$ and $f_{\mathrm{R}}^{(\mathrm{d})}$. The additional force may possibly be applied at the node (e), which is illustrated as $\boldsymbol{f}_{\mathrm{R}}^{(\mathrm{e})}$ in Figure 2. Such an uncertainty model can be written as $\boldsymbol{f}_{\mathrm{R}}\in\mathcal{R}_p(\alpha,\tilde{\boldsymbol{f}}_{\mathrm{R}})$ with

$$\mathcal{R}_p(\alpha,\tilde{\boldsymbol{f}}_{\mathrm{R}})=\left\{\boldsymbol{f}_{\mathrm{R}}\in\Re^{n^{\mathrm{d}}}\,\middle|\,\boldsymbol{f}_{\mathrm{R}}=\tilde{\boldsymbol{f}}_{\mathrm{R}}+F_0\boldsymbol{\zeta},\quad\alpha\geq\|\boldsymbol{\zeta}\|_p,\quad\sum_{i=1}^{n^{\mathrm{z}}}\zeta_i=0\right\}.\tag{26}$$

Here, F_0 is assumed to satisfy the condition that the components of the vector $F_0\,\boldsymbol{\zeta}$ corresponding to the directions of $\tilde{\boldsymbol{f}}_{\mathrm{R}}$ only are possibly not equal to zeros for any $\boldsymbol{\zeta}\in\Re^{n^{\mathrm{z}}}$ as shown in Figure 2. Note that $n^{\mathrm{z}}=3$ and $\mathrm{rank}(F_0)=3$ in the example of Figure 2. The condition $\sum_{i=1}^{n^{\mathrm{z}}}\zeta_i=0$ in (26) is added in order to normalize the magnitude of $\boldsymbol{f}_{\mathrm{R}}$. We can easily see that the uncertainty set $\mathcal{D}_p$ defined by Equation (26) satisfies the nesting and contraction axioms of the info-gap uncertainty model introduced in Section 2.2.

Let $p=2$ in the uncertainty model (26). Then the following proposition implies that the robustness function defined by (25) is obtained as the optimal value of an LP problem.

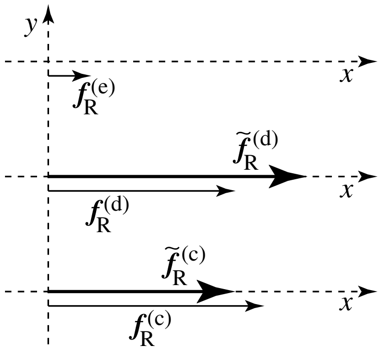

Figure 2. Uncertainty of distribution of the reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$.

Proposition 3.3. Define $\boldsymbol{n} \in \Re^{n^{\mathrm{z}}}$ by

$$\boldsymbol{n} = \frac{1}{\sqrt{n^{\mathrm{z}}}}(1, 1, \dots, 1)^{\top}. \tag{27}$$

Then the robustness function, $\hat{\alpha}(\underline{\lambda})$ defined by (25) and (26) with $p = 2$, is obtained as the optimal value of the LP problem

$$\begin{aligned}\hat{\alpha}(\underline{\lambda}) = \max_{r,\boldsymbol{\xi}} \Big\{ r \,\Big|\, & \boldsymbol{h}_i^{\dagger}(\underline{\lambda}\tilde{\boldsymbol{f}}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}) + r\sqrt{\|\boldsymbol{h}_i^{\dagger}F_0\|_2^2 - (\boldsymbol{h}_i^{\dagger}F_0\boldsymbol{n})^2} + \boldsymbol{h}_i^{\perp}\boldsymbol{\xi} \le \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}}, \\ & -\boldsymbol{h}_i^{\dagger}(\underline{\lambda}\tilde{\boldsymbol{f}}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}) + r\sqrt{\|\boldsymbol{h}_i^{\dagger}F_0\|_2^2 - (\boldsymbol{h}_i^{\dagger}F_0\boldsymbol{n})^2} - \boldsymbol{h}_i^{\perp}\boldsymbol{\xi} \le \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}} \Big\}\end{aligned} \tag{28}$$

in the variables $r \in \Re$ and $\boldsymbol{\xi} \in \Re^{n^{\xi}}$.

Proof. In a manner similar to Proposition 3.1, we can show that

$$\hat{\alpha}(\underline{\lambda}) = \max_{r,\boldsymbol{q},\boldsymbol{\zeta}} \left\{ r \,\middle|\, H\boldsymbol{q} = \underline{\lambda}\tilde{\boldsymbol{f}}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}} + \underline{\lambda}F_0\boldsymbol{\zeta}, \quad \boldsymbol{q} \in \mathcal{Q} \quad (\text{for all } \boldsymbol{\zeta} \in \mathcal{B}_p(r)) \right\} \tag{29}$$

holds, because we may replace $F_0\boldsymbol{\zeta}$ with $\underline{\lambda}F_0\boldsymbol{\zeta}$ in the proof of Proposition 3.1. In a manner similar to Equation (22), we see that the constraints of the problem (29) are equivalently rewritten as

$$\boldsymbol{h}_i^{\dagger}(\underline{\lambda}\tilde{\boldsymbol{f}}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}) + \boldsymbol{h}_i^{\perp}\boldsymbol{\xi} + q_i^{\max} \le \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}}, \tag{30}$$

$$\boldsymbol{h}_i^{\dagger}(\underline{\lambda}\tilde{\boldsymbol{f}}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}) + \boldsymbol{h}_i^{\perp}\boldsymbol{\xi} + q_i^{\min} \le \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}}, \tag{31}$$

where

$$q_i^{\max} := \max_{\boldsymbol{\zeta}} \left\{ \boldsymbol{h}_i^{\dagger}F_0\boldsymbol{\zeta} \,\middle|\, r \ge \|\boldsymbol{\zeta}\|_2, \ \sum_{i=1}^{n^{\mathrm{z}}} \zeta_i = 0 \right\}, \tag{32}$$

$$q_i^{\min} := \min_{\boldsymbol{\zeta}} \left\{ \boldsymbol{h}_i^{\dagger}F_0\boldsymbol{\zeta} \,\middle|\, r \ge \|\boldsymbol{\zeta}\|_2, \ \sum_{i=1}^{n^{\mathrm{z}}} \zeta_i = 0 \right\} = -q_i^{\max}. \tag{33}$$

Analogous to the key equation, (23), in the proof of Proposition 3.2, we next evaluate $q_i^{\max}$ defined by (32) analytically. Note that $\boldsymbol{n}$ defined in Proposition 3.3 corresponds to the unit normal vector of the hyperplane

$$\pi = \left\{ \boldsymbol{\zeta} \in \Re^{n^{\mathrm{z}}} \;\middle|\; \sum_{i=1}^{n^{\mathrm{z}}} \zeta_i = 0 \right\}.$$

Define $\boldsymbol{b}_i \in \Re^{n^{\mathrm{z}}}$ by $\boldsymbol{b}_i = (\boldsymbol{h}_i^{\dagger} F_0)^{\top}$ for simplicity. Let $\boldsymbol{w}_i \in \Re^{n^{\mathrm{z}}}$ denote the projection of the vector $\boldsymbol{b}_i$ onto π, which is written as

$$\begin{aligned} \boldsymbol{w}_i &= \boldsymbol{b}_i - (\boldsymbol{b}_i^{\top}\boldsymbol{n})\boldsymbol{n} \\ &= (\boldsymbol{h}_i^{\dagger} F_0)^{\top} - (\boldsymbol{h}_i^{\dagger} F_0 \boldsymbol{n})\boldsymbol{n}. \end{aligned} \tag{34}$$

Since $\|\boldsymbol{n}\|_2 = 1$, we obtain

$$\|\boldsymbol{w}_i\|_2^2 = \|\boldsymbol{b}_i\|_2^2 - (\boldsymbol{b}_i^{\top}\boldsymbol{n})^2. \tag{35}$$

Then $q_i^{\max}$ in Equation (32) is written as

$$q_i^{\max} = \max_{\boldsymbol{\zeta}} \left\{ \boldsymbol{b}_i^{\top}\boldsymbol{\zeta} \;\middle|\; r \geq \|\boldsymbol{\zeta}\|_2,\ \sum_{i=1}^{n^{\mathrm{z}}} \zeta_i = 0 \right\} = \max_{\boldsymbol{\zeta}} \left\{ \boldsymbol{w}_i^{\top}\boldsymbol{\zeta} \mid r \geq \|\boldsymbol{\zeta}\|_2,\ \boldsymbol{\zeta} \in \pi \right\}.$$

Since both $\boldsymbol{w}_i$ and $\boldsymbol{\zeta}_i$ are on the hyperplane π, we obtain

$$q_i^{\max} = \max_{\boldsymbol{\zeta}} \{ \|\boldsymbol{w}_i\|_2 \|\boldsymbol{\zeta}\|_2 \mid r \geq \|\boldsymbol{\zeta}\|_2 \} = r\|\boldsymbol{w}_i\|_2. \tag{36}$$

By using Equation (35), we see that (36) is rewritten as

$$q_i^{\max} = r\sqrt{\|\boldsymbol{b}_i\|_2^2 - (\boldsymbol{b}_i^{\top}\boldsymbol{n})^2}. \tag{37}$$

By substituting Equation (37) into (30) and (31), and by using (29), we obtain (28). □

The following proposition provides an LP problem to compute the robustness function (25) in the case of $p = \infty$ in Equation (26).

Proposition 3.4. Let l_i be

$$l_i = \max_{j \in \{1,\dots,n^{\mathrm{z}}\}} |w_{ij}|, \quad i = 1, \dots, n^{\mathrm{m}},$$

where the vector $\boldsymbol{w}_i = (w_{ij}) \in \Re^{n^{\mathrm{z}}}$ is defined by Equation (34). Then the robustness function $\hat{\alpha}(\underline{\lambda})$ defined by (25) and (26) with $p = \infty$ is obtained as the optimal value of the LP problem

$$\begin{aligned} \hat{\alpha}(\underline{\lambda}) = \max_{r,\boldsymbol{\xi}} \Big\{ r \;\Big|\; & \boldsymbol{h}_i^{\dagger}(\underline{\lambda}\tilde{\boldsymbol{f}}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}) + r\left[\|\boldsymbol{h}_i^{\dagger}F_0\|_2^2 - (\boldsymbol{h}_i^{\dagger}F_0\boldsymbol{n})^2\right]/l_i + \boldsymbol{h}_i^{\perp}\boldsymbol{\xi} \leq \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}}, \\ & -\boldsymbol{h}_i^{\dagger}(\underline{\lambda}\tilde{\boldsymbol{f}}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}) + r\left[\|\boldsymbol{h}_i^{\dagger}F_0\|_2^2 - (\boldsymbol{h}_i^{\dagger}F_0\boldsymbol{n})^2\right]/l_i - \boldsymbol{h}_i^{\perp}\boldsymbol{\xi} \leq \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}} \Big\} \end{aligned} \tag{38}$$

in the variables $r \in \Re$ and $\boldsymbol{\xi} \in \Re^{n^{\xi}}$.

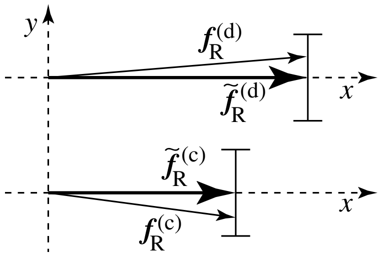

Figure 3. Uncertainty of direction of the reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$.

Proof. The assertion can be shown in a manner similar to Proposition 3.3. Particularly, we should evaluate

$$q_i^{\max,\infty} := \max_{\zeta} \left\{ \boldsymbol{h}_i^{\dagger} F_0 \boldsymbol{\zeta} \;\middle|\; r \geq \|\boldsymbol{\zeta}\|_{\infty}, \quad \sum_{i=1}^{n^{\mathrm{z}}} \zeta_i = 0 \right\} \tag{39}$$

instead of Equation (32). Observe that

$$\max_{\zeta,\beta} \{ \|\boldsymbol{\zeta}\|_2 \mid r \geq \|\boldsymbol{\zeta}\|_{\infty}, \quad \boldsymbol{\zeta} = \beta \boldsymbol{w}_i \} = \frac{1}{l_i} \|\boldsymbol{w}_i\|_2,$$

from which we see that Equation (39) is reduced to

$$\begin{aligned} q_i^{\max,\infty} &= \max_{\zeta} \left\{ \boldsymbol{b}_i^{\top} \boldsymbol{\zeta} \;\middle|\; r \geq \|\boldsymbol{\zeta}\|_{\infty}, \quad \sum_{i=1}^{n^{\mathrm{z}}} \zeta_i = 0 \right\} \\ &= \max_{\zeta} \left\{ \boldsymbol{w}_i^{\top} \boldsymbol{\zeta} \mid r \geq \|\boldsymbol{\zeta}\|_{\infty}, \quad \boldsymbol{\zeta} \in \pi \right\} \\ &= \max_{\zeta,\beta} \{ \|\boldsymbol{w}_i\|_2 \|\boldsymbol{\zeta}\|_2 \mid r \geq \|\boldsymbol{\zeta}\|_{\infty}, \quad \boldsymbol{\zeta} = \beta \boldsymbol{w}_i \} \\ &= \frac{r}{l_i} \|\boldsymbol{w}_i\|_2^2. \end{aligned} \tag{40}$$

By substituting (34) and (40) into (30) and (31), we obtain the constraints of the problem (38), which concludes the proof. □

3.3.2. *Uncertainty of load direction.* Recall the example of a truss illustrated in Figure 1. To apply the reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$, external forces are applied at the nodes (c) and (d), which are denoted by $\boldsymbol{f}_{\mathrm{R}}^{(\mathrm{c})}$ and $\boldsymbol{f}_{\mathrm{R}}^{(\mathrm{d})}$, respectively. Suppose that the directions of $\boldsymbol{f}_{\mathrm{R}}^{(\mathrm{c})}$ and $\boldsymbol{f}_{\mathrm{R}}^{(\mathrm{d})}$ are uncertain as illustrated in Figure 3. Such an uncertainty model can be realized as $\boldsymbol{f}_{\mathrm{R}} \in \mathcal{R}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{R}})$ with

$$\mathcal{R}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{R}}) = \left\{ \boldsymbol{f}_{\mathrm{R}} \in \Re^{n^{\mathrm{d}}} \;\middle|\; \boldsymbol{f}_{\mathrm{R}} = \tilde{\boldsymbol{f}}_{\mathrm{R}} + F_0 \boldsymbol{\zeta}, \quad \alpha \geq \|\boldsymbol{\zeta}\|_p \right\}. \tag{41}$$

Here, F_0 is assumed to satisfy the condition that the components of $F_0 \boldsymbol{\zeta}$ corresponding only to the directions orthogonal to the $\tilde{\boldsymbol{f}}_{\mathrm{R}}$ are possibly not equal to zeros as illustrated in Figure 3. Note that $n^{\mathrm{z}} = 2$ and $\mathrm{rank}(F_0) = 2$ in the case of Figure 3.

In this section, we assume that the magnitude of uncertainty α in the uncertainty model (41) of $\boldsymbol{f}_{\mathrm{R}}$ is sufficiently small. This is because the magnitude of the reference load $\boldsymbol{f}_{\mathrm{R}}$ varies in (41). Obviously, formulations presented below are valid for arbitrary large value of α. However, from the engineering view point, we restrict ourselves to the case in which the variation of the magnitude of $\boldsymbol{f}_{\mathrm{R}}$ does not cause the ambiguity of the definition of the limit load factor when $\boldsymbol{f}_{\mathrm{R}}$ is running through $\mathcal{R}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{R}})$ defined by (41). Under this assumption, the definition (25) of the robustness function is guaranteed to be a proper measure of robustness. On the other hand, if this assumption is not satisfied, then the constraint $\lambda(\boldsymbol{f}_{\mathrm{R}}, \tilde{\boldsymbol{f}}_{\mathrm{D}}) \geq \underline{\lambda}$ does not have a proper meaning. To date, it is not clear whether the robustness function can be reformulated into a tractable form or not when we add the condition of normalization to the magnitude of $\boldsymbol{f}_{\mathrm{R}}$ in Equation (41). Instead, we can show that the robustness function is computed easily without a normalization condition, which is the contribution of this section. Note again that all results other than those in this section are valid for arbitrary large magnitude α of uncertainties. Particularly, in Section 3.3.1, it should be emphasized that we have considered the normalization condition of $\boldsymbol{f}_{\mathrm{R}}$ in (26).

The following proposition is obtained easily in a manner similar to Proposition 3.2.

Proposition 3.5. The robustness function $\hat{\alpha}(\underline{\lambda})$ defined by (25) and (41) is obtained as the optimal value of the LP problem

$$\begin{aligned}\hat{\alpha}(\underline{\lambda}) = \max_{r,\boldsymbol{\xi}} \Big\{ r \,\Big|\, & \boldsymbol{h}_i^{\dagger}(\underline{\lambda}\tilde{\boldsymbol{f}}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}) + r\underline{\lambda}\|\boldsymbol{h}_i^{\dagger} F_0\|_{p^*} + \boldsymbol{h}_i^{\perp}\boldsymbol{\xi} \leq \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}}, \\ & -\boldsymbol{h}_i^{\dagger}(\underline{\lambda}\tilde{\boldsymbol{f}}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}) + r\underline{\lambda}\|\boldsymbol{h}_i^{\dagger} F_0\|_{p^*} - \boldsymbol{h}_i^{\perp}\boldsymbol{\xi} \leq \bar{q}_i, \quad i = 1, \dots, n^{\mathrm{m}} \Big\}\end{aligned} \tag{42}$$

in the variables $r \in \Re$ and $\boldsymbol{\xi} \in \Re^{n^{\xi}}$.

4. Robustness analysis of framed structures

4.1. ***Basic problem of limit analysis of frames.*** Consider a framed structure in the two-dimensional space that consists of a finite number of the conventional Euler–Bernoulli beam elements. The number of elements is denoted by n^{m}. Let q_i denote the axial force of the ith member. The moments acting on two endpoints are denoted by m_i^1 and m_i^2. The shear force at the endpoint of the ith member is denoted by τ_i.

The equilibrium equations between the external force $(\lambda \boldsymbol{f}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}})$ and the internal forces $\boldsymbol{q}$, $\boldsymbol{m}^1$, $\boldsymbol{m}^2$, and $\boldsymbol{\tau}$ can be written in the form of

$$H^q \boldsymbol{q} + H_1^m \boldsymbol{m}^1 + H_2^m \boldsymbol{m}^2 + H^{\tau} \boldsymbol{\tau} = \lambda \boldsymbol{f}_{\mathrm{R}} + \boldsymbol{f}_{\mathrm{D}}, \tag{43}$$

where $\boldsymbol{q} = (q_i) \in \Re^{n^{\mathrm{m}}}$, $\boldsymbol{m}^1 = (m_i^1) \in \Re^{n^{\mathrm{m}}}$, $\boldsymbol{m}^2 = (m_i^2) \in \Re^{n^{\mathrm{m}}}$, and $\boldsymbol{\tau} = (\tau_i) \in \Re^{n^{\mathrm{m}}}$. The matrices H^q, H_1^m, H_2^m, and $H^{\tau} \in \Re^{n^{\mathrm{d}} \times n^{\mathrm{m}}}$ are constant matrices, where n^{d} denotes the number of degrees of freedom of displacements of the frame. The moment equilibria of internal forces are written as

$$l_i \tau_i + m_i^1 + m_i^2 = 0, \quad i = 1, \dots, n^{\mathrm{m}}. \tag{44}$$

By letting $\boldsymbol{y} = (\boldsymbol{q}, \boldsymbol{m}^1, \boldsymbol{m}^2, \boldsymbol{\tau}) \in \Re^{n^{\mathrm{m}}} \times \Re^{n^{\mathrm{m}}} \times \Re^{n^{\mathrm{m}}} \times \Re^{n^{\mathrm{m}}}$ for simplicity, the equilibrium equations (43) and (44) can be condensed in the form of

$$H\boldsymbol{y} = \lambda \boldsymbol{f}_{\mathrm{R}}^0 + \boldsymbol{f}_{\mathrm{D}}^0, \tag{45}$$

where H is a constant matrix. Here, the constant vector $\boldsymbol{f}_{\mathrm{R}}^0$ consists of the components of $\boldsymbol{f}_{\mathrm{R}}$ and $\boldsymbol{0}$, while $\boldsymbol{f}_{\mathrm{D}}^0$ consists of the components of $\boldsymbol{f}_{\mathrm{D}}$ and $\boldsymbol{0}$.

We next introduce the yielding condition of a beam element. Suppose that the members experience plastic deformations only at their two ends. Provided that the dependence of the yield condition on the shear force is negligible, the admissible set of internal forces is given as

$$\mathcal{Y} = \left\{ \boldsymbol{y} = (\boldsymbol{q}, \boldsymbol{m}^1, \boldsymbol{m}^2, \boldsymbol{\tau}) \;\middle|\; \frac{|q_i + q_i^{\mathrm{p}}|}{\bar{q}_i} + \frac{|m_i^j|}{\bar{m}_i} \leq 1 \quad (\text{for all } i \in \{1, \dots, n^{\mathrm{m}}\}, \ \text{for all } j \in \{1, 2\}) \right\}, \tag{46}$$

where the set $\mathcal{Y}$ is illustrated in Figure 4. Here, $\bar{q}_i$, $\bar{m}_i$, and q_i^{p} are given constants.

From (45) and (46), the limit load factor for the fixed $\boldsymbol{f}_{\mathrm{R}}$ and $\boldsymbol{f}_{\mathrm{D}}$ is obtained by solving the following LP problem:

$$\lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}}) = \max_{\lambda, \boldsymbol{y}} \left\{ \lambda \;\middle|\; H\boldsymbol{y} = \lambda \boldsymbol{f}_{\mathrm{R}}^0 + \boldsymbol{f}_{\mathrm{D}}^0, \quad \boldsymbol{y} \in \mathcal{Y} \right\}. \tag{47}$$

4.2. ***Robustness function under uncertain dead load.*** In a manner similar to a truss investigated in Section 3, we can formulate LP problems providing the robustness function under various uncertainty models of the dead load $\boldsymbol{f}_{\mathrm{D}}$ and the reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$. For simplicity of the presentation, we pay attention only to the uncertainty model (10) of $\boldsymbol{f}_{\mathrm{D}}$. It is straightforward to extend the result below to the other uncertainty models investigated in Section 3.3. The remainder of this section is devoted to reformulating the problem (11) for framed structures into a numerically tractable problem.

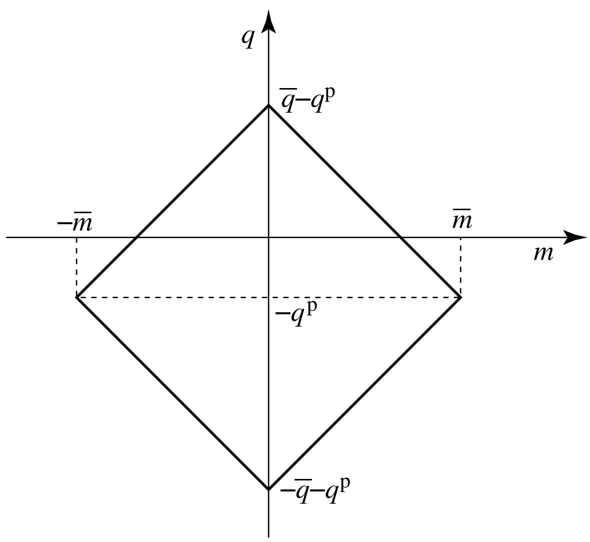

Figure 4. Yielding surface of the beam element.

Analogous to (18) in the proof of Proposition 3.2, any $\boldsymbol{y}$ that solves the system of linear equations (45) can be written as

$$\boldsymbol{y} = H^{\dagger}(\lambda \boldsymbol{f}_{\mathrm{R}}^{0} + \tilde{\boldsymbol{f}}_{\mathrm{D}}^{0}) + (H^{\dagger} F_0)\boldsymbol{\zeta} + H^{\perp}\boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \Re^{n^{\xi}}. \tag{48}$$

For simplicity, we write Equation (48) component-wise as

$$q_i = \frac{1}{\bar{q}_i}\left[\boldsymbol{h}_i^q(\underline{\lambda} \boldsymbol{f}_{\mathrm{R}}^{0} + \tilde{\boldsymbol{f}}_{\mathrm{D}}^{0}) + (\boldsymbol{h}_i^q F_0)\boldsymbol{\zeta} + \bar{\boldsymbol{h}}_i^q \boldsymbol{\xi}\right], \qquad i = 1, \dots, n^{\mathrm{m}}, \tag{49}$$

$$m_i^1 = \frac{1}{\bar{m}_i}\left[\boldsymbol{h}_{i,1}^m(\underline{\lambda} \boldsymbol{f}_{\mathrm{R}}^{0} + \tilde{\boldsymbol{f}}_{\mathrm{D}}^{0}) + (\boldsymbol{h}_{i,1}^m F_0)\boldsymbol{\zeta} + \bar{\boldsymbol{h}}_{i,1}^m \boldsymbol{\xi}\right], \quad i = 1, \dots, n^{\mathrm{m}}, \tag{50}$$

$$m_i^2 = \frac{1}{\bar{m}_i}\left[\boldsymbol{h}_{i,2}^m(\underline{\lambda} \boldsymbol{f}_{\mathrm{R}}^{0} + \tilde{\boldsymbol{f}}_{\mathrm{D}}^{0}) + (\boldsymbol{h}_{i,2}^m F_0)\boldsymbol{\zeta} + \bar{\boldsymbol{h}}_{i,2}^m \boldsymbol{\xi}\right], \quad i = 1, \dots, n^{\mathrm{m}}, \tag{51}$$

where $\boldsymbol{h}_i^q/\bar{q}_i$, $\boldsymbol{h}_{i,1}^m/\bar{m}_i$, and $\boldsymbol{h}_{i,2}^m/\bar{m}_i$ denote appropriate row vectors of $H^{\dagger}$. Similarly, $\bar{\boldsymbol{h}}_i^q/\bar{q}_i$, $\bar{\boldsymbol{h}}_{i,1}^m/\bar{m}_i$, and $\bar{\boldsymbol{h}}_{i,2}^m/\bar{m}_i$ correspond to appropriate row vectors of $H^{\perp}$.

Proposition 4.1. For a framed structure with the yield condition (46), the robustness function $\hat{\alpha}(\underline{\lambda})$ defined by (10) and (11) is obtained as the optimal value of the LP problem

$$\begin{aligned}\hat{\alpha}(\underline{\lambda}) = \max_{r,\boldsymbol{\xi}} \Big\{ r \,\Big|\, & \left[(-1)^{\mu}\boldsymbol{h}_i^q/\bar{q}_i + (-1)^{\nu}\boldsymbol{h}_{i,j}^m/\bar{m}_i\right](\underline{\lambda} \boldsymbol{f}_{\mathrm{R}}^{0} + \tilde{\boldsymbol{f}}_{\mathrm{D}}^{0}) + (-1)^{\mu} q_i^{\mathrm{p}}/\bar{q}_i \\ & + r\left\|\left[(-1)^{\mu}\boldsymbol{h}_i^q/\bar{q}_i + (-1)^{\nu}\boldsymbol{h}_{i,j}^m/\bar{m}_i\right]F_0\right\|_{p^*} + \left[(-1)^{\mu}\bar{\boldsymbol{h}}_i^q/\bar{q}_i + (-1)^{\nu}\bar{\boldsymbol{h}}_{i,j}^m/\bar{m}_i\right]\boldsymbol{\xi} \le 1, \\ & i = 1, \dots, n^m, \ (j, \mu, \nu) \in \{1, 2\}^3 \Big\}\end{aligned} \tag{52}$$

in the variables $r \in \Re$ and $\boldsymbol{\xi} \in \Re^{n^{\xi}}$.

Proof. In a manner similar to Proposition 3.1, we can show that $\hat{\alpha}(\underline{\lambda})$ is obtained as

$$\hat{\alpha}(\underline{\lambda}) = \max_{r,\boldsymbol{y}} \left\{ r \,\middle|\, H\boldsymbol{y} = \underline{\lambda} \boldsymbol{f}_{\mathrm{R}}^{0} + \tilde{\boldsymbol{f}}_{\mathrm{D}}^{0} + F_0\boldsymbol{\zeta}, \quad \boldsymbol{y} \in \mathcal{Y} \quad (\text{for all } \boldsymbol{\zeta} \in \mathcal{B}_p(r)) \right\}. \tag{53}$$

Analogous to the proof of Proposition 3.2, we investigate the constraints of the problem (53). Since q_i, m_i^1, and m_i^2 in (49)–(51) are regarded as linear functions of $\boldsymbol{\zeta}$, we write $q_i(\boldsymbol{\zeta})$, $m_i^1(\boldsymbol{\zeta})$, and $m_i^2(\boldsymbol{\zeta})$ for simplicity. By using the definition (46) of $\mathcal{Y}$, the constraints of (53) can be rewritten as

$$\max_{\zeta}\left\{\frac{q_i(\boldsymbol{\zeta})-q_i^{\mathrm{p}}}{\bar{q}_i}+\frac{m_i^j(\boldsymbol{\zeta})}{\bar{m}_i}\,\middle|\,\boldsymbol{\zeta}\in\mathcal{B}_p(r)\right\}\le 1,$$
$$\max_{\zeta}\left\{\frac{q_i(\boldsymbol{\zeta})-q_i^{\mathrm{p}}}{\bar{q}_i}-\frac{m_i^j(\boldsymbol{\zeta})}{\bar{m}_i}\,\middle|\,\boldsymbol{\zeta}\in\mathcal{B}_p(r)\right\}\le 1,$$
$$\max_{\zeta}\left\{-\frac{q_i(\boldsymbol{\zeta})-q_i^{\mathrm{p}}}{\bar{q}_i}+\frac{m_i^j(\boldsymbol{\zeta})}{\bar{m}_i}\,\middle|\,\boldsymbol{\zeta}\in\mathcal{B}_p(r)\right\}\le 1,$$
$$\max_{\zeta}\left\{-\frac{q_i(\boldsymbol{\zeta})-q_i^{\mathrm{p}}}{\bar{q}_i}-\frac{m_i^j(\boldsymbol{\zeta})}{\bar{m}_i}\,\middle|\,\boldsymbol{\zeta}\in\mathcal{B}_p(r)\right\}\le 1,\quad \text{for all } i\in\{1,\dots,n^m\},\ j\in\{1,2\},$$

which are simply written as

$$\max_{\zeta}\left\{(-1)^{\mu}\frac{q_i(\boldsymbol{\zeta})-q_i^{\mathrm{p}}}{\bar{q}_i}+(-1)^{\nu}\frac{m_i^j(\boldsymbol{\zeta})}{\bar{m}_i}\,\middle|\,r\ge\|\zeta\|_p\right\}\le 1, \tag{54}$$
$$\text{for all } i\in\{1,\dots,n^m\},\quad j\in\{1,2\},\quad \mu\in\{1,2\},\quad \nu\in\{1,2\}. \tag{55}$$

From the Hölder inequality [Michael Sttele 2004], the equation

$$\max_{\zeta}\left\{\boldsymbol{b}^{\top}\boldsymbol{\zeta}\,\middle|\,r\ge\|\boldsymbol{\zeta}\|_p\right\}=r\|\boldsymbol{b}\|_{p^*}$$

holds for any constant $\boldsymbol{b}\in\Re^k$, from which it follows that Equation (55) is equivalent to

$$\begin{aligned}\Big[(-1)^{\mu}\boldsymbol{h}_i^q/\bar{q}_i+(-1)^{\nu}\boldsymbol{h}_{i,j}^m/\bar{m}_i\Big]&(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}}^0+\tilde{\boldsymbol{f}}_{\mathrm{D}}^0)+(-1)^{\mu}q_i^{\mathrm{p}}/\bar{q}_i\\ &+r\left\|\Big[(-1)^{\mu}\boldsymbol{h}_i^q/\bar{q}_i+(-1)^{\nu}\boldsymbol{h}_{i,j}^m/\bar{m}_i\Big]F_0\right\|_{p^*}+\Big[(-1)^{\mu}\bar{\boldsymbol{h}}_i^q/\bar{q}_i+(-1)^{\nu}\bar{\boldsymbol{h}}_{i,j}^m/\bar{m}_i\Big]\\ &\boldsymbol{\xi}\le 1,\ \text{for all } i\in\{1,\dots,n^{\mathrm{m}}\},\quad j\in\{1,2\},\quad \mu\in\{1,2\},\quad \nu\in\{1,2\}.\end{aligned} \tag{56}$$

Substitution of Equation (56) into (53) results in (52), which concludes the proof. □

Proposition 4.1 is important, because it implies that the robustness function of a framed structure can be obtained easily by solving the LP problem (52). Similarly, it can be shown that the robustness function with respect to the uncertain reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$ is also obtained as the optimal value of an LP problem, provided that $\boldsymbol{f}_{\mathrm{R}}$ obeys the uncertainty model introduced in Section 3.3.

5. Numerical experiments

In the following examples, computation was carried out on a Pentium M (1.7 GHz with 512 MB memory) with MATLAB V. 7.3 [MatLab 2006]. We solve an LP problem by using the MATLAB built-in function `linprog`. In the following examples, we mainly consider the uncertainty (10) of the dead load $\boldsymbol{f}_{\mathrm{D}}$ with $p=2$ in order to avoid the redundancy of presentation. However, it should be emphasized that our major contribution of this paper is to present the LP reformulation of the info-gap robustness function under various uncertainty models of $\boldsymbol{f}_{\rm D}$ and $\boldsymbol{f}_{\rm R}$.

5.1. 3 × 3 *truss*. Consider a plane truss illustrated in Figure 1, where $W = 70.0$ cm, $H = 50.0$ cm, $n^{\rm d} = 28$, and $n^{\rm m} = 42$. The nodes (a) and (b) are pin-supported.

As the nominal dead load $\tilde{\boldsymbol{f}}_{\rm D}$, we apply the external forces $(0, -300.0)$ kN at the nodes (f) and (g) as shown in Figure 1. Note again that $\boldsymbol{f}_{\rm D}$ represents the sum of conventional live load and dead load in civil engineering. The nominal reference disturbance load $\tilde{\boldsymbol{f}}_{\rm R}$ is defined such that $(100.0, 0)$ kN and $(50.0, 0)$ kN, respectively, are applied at the nodes (c) and (d). For each member, the yield stress is $\sigma_i^{\rm y} = 400$ MPa and cross-sectional area is $a_i = 25.0\,{\rm cm}^2$ in Equation (7). Note that this example is similar to the example investigated in [Kanno and Takewaki 2007] for computing the worst-case limit load factor under the uncertainty of dead load.

The limit load factor under the nominal load is computed as $\lambda^*(\tilde{\boldsymbol{f}}_{\rm R}, \tilde{\boldsymbol{f}}_{\rm D}) = 24.18$ by employing the usual limit analysis, that is, by solving the LP problem (9). The collapse mode corresponds to the sway-type with horizontal displacements of the joints as shown in Figure 5.

Firstly, suppose that the dead load $\boldsymbol{f}_{\rm D}$ obeys the uncertainty model (10), while $\boldsymbol{f}_{\rm R}$ is assumed to be certain. Consider the following two cases:

$$\text{Case 1:} \quad p = 2 \quad \text{in Equation (10);}$$
$$\text{Case 2:} \quad p = \infty \quad \text{in Equation (10).}$$

The uncertain dead load $F_0\,\boldsymbol{\zeta}$ is assumed to exist possibly at all free nodes except for the nodes (c) and (d). At the nodes (f) and (g), the uncertain load is supposed to exist in the directions orthogonal to $\tilde{\boldsymbol{f}}_{\rm D}$. Hence, the reference disturbance load is guaranteed to be unchanged as discussed in Section 3.2, where $n^{\rm z} = 22$ in Equation (10). The coefficient matrix F_0 is defined so that the uncertainties of components of the vector $F_0\,\boldsymbol{\zeta}$ have no correlation, and each nonzero component of F_0 is equal to 100.0 kN. Accordingly, in Case 2, the uncertain load $F_0\,\boldsymbol{\zeta}$ runs through the squares and arrows depicted with the dotted lines in Figure 1. We set $\underline{\lambda} = 23.0$ in Equation (11). By solving the LP problem (17), we obtain $\hat{\alpha}(\underline{\lambda}) = 0.4170$ in Case 1 and $\hat{\alpha}(\underline{\lambda}) = 0.1000$ in Case 2.

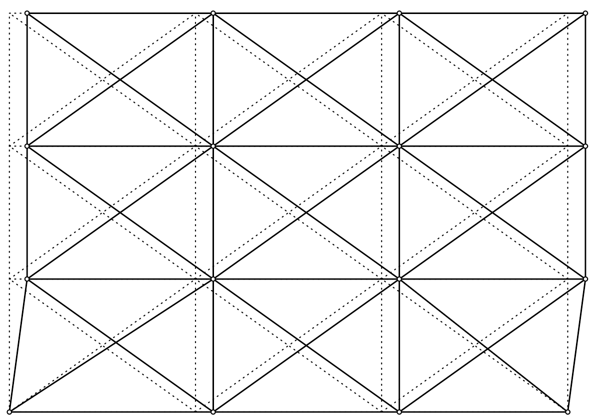

Figure 5. Collapse mode of 3 × 3 truss with the nominal external load.

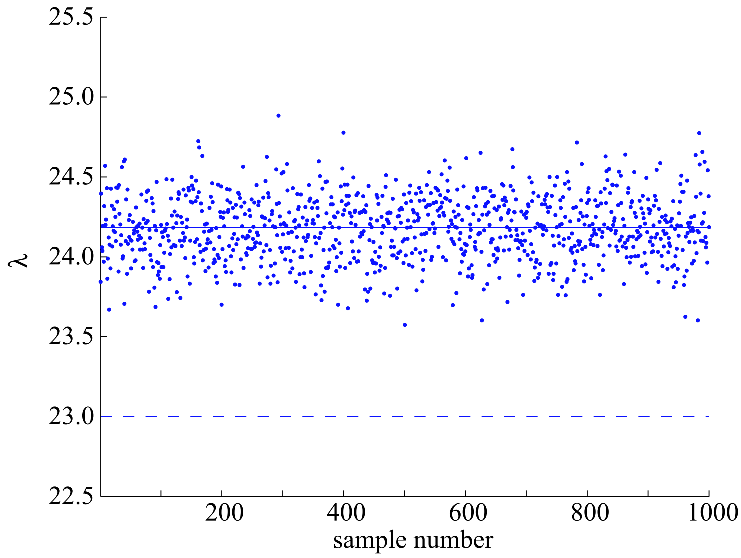

Figure 6. Limit load factor of the 3×3 truss in Case 1 for the uncertain $\boldsymbol{f}_{\rm D}$ corresponding to randomly generated $\boldsymbol{\zeta}$ (— nominal limit load factor $\lambda^*(\tilde{\boldsymbol{f}}_{\rm R}, \tilde{\boldsymbol{f}}_{\rm D})$; – – performance requirement $\underline{\lambda}$.

For Case 1, we randomly generate a number of $\boldsymbol{\zeta}$ satisfying $\|\boldsymbol{\zeta}\|_2 = \hat{\alpha} = 0.4170$, and perform the limit analysis. The limit load factors $\lambda^*(\tilde{\boldsymbol{f}}_{\rm R}, \boldsymbol{f}_{\rm D}(\boldsymbol{\zeta}))$ obtained are shown in Figure 6 as many points. It is observed from Figure 6 that all generated limit load factors are larger than the lower bound $\underline{\lambda}$, which supports the assertion that the constraint $\lambda^*(\tilde{\boldsymbol{f}}_{\rm R}, \boldsymbol{f}_{\rm D}) \geq \underline{\lambda}$ is guaranteed to be satisfied for any $\boldsymbol{f}_{\rm D} \in \mathcal{D}_2(\hat{\alpha}, \tilde{\boldsymbol{f}}_{\rm D})$. Note that the actual worst-case dead load cannot be exactly predicted, in general, by taking a rather small number of random samples of $\boldsymbol{\zeta}$. Hence, in Figure 6 we cannot find the case in which the limit load factor coincides with $\underline{\lambda}$.

Figure 7 depicts the variation of the robustness function $\hat{\alpha}$ with respect to the performance requirement $\underline{\lambda}$. It is observed from Figure 7 that $\hat{\alpha} = 0$ corresponds to $\underline{\lambda} = \lambda^*(\tilde{\boldsymbol{f}}_{\rm R}, \tilde{\boldsymbol{f}}_{\rm D})$, that is, the robustness function vanishes if $\underline{\lambda}$ is equal to the nominal limit load factor. The variation of $\hat{\alpha}$ possesses an angular point. This is because the worst-case dead load as well as the collapse mode in the worst case depends on the magnitude of uncertainty as observed in [Kanno and Takewaki 2007, section 6.1].

We next investigate robustness of the truss against the uncertain reference disturbance load $\boldsymbol{f}_{\rm R}$ as discussed in Section 3.3. The dead load $\boldsymbol{f}_{\rm D}$ is supposed to be certain. Consider the following three cases:

$$\begin{aligned} &\text{Case 3:} \quad p = 2 \quad \text{in Equation (26);} \\ &\text{Case 4:} \quad p = \infty \quad \text{in Equation (26);} \\ &\text{Case 5:} \quad p = 2 \quad \text{in Equation (41).} \end{aligned}$$

Note that $n^{\rm z} = 3$ in Case 3 and Case 4 as illustrated in Figure 2, while $n^{\rm z} = 2$ in Case 5 as illustrated in Figure 3. Each nonzero component of F_0 is equal to 100.0 kN. We set $\underline{\lambda} = 23.0$ in Equation (25). The

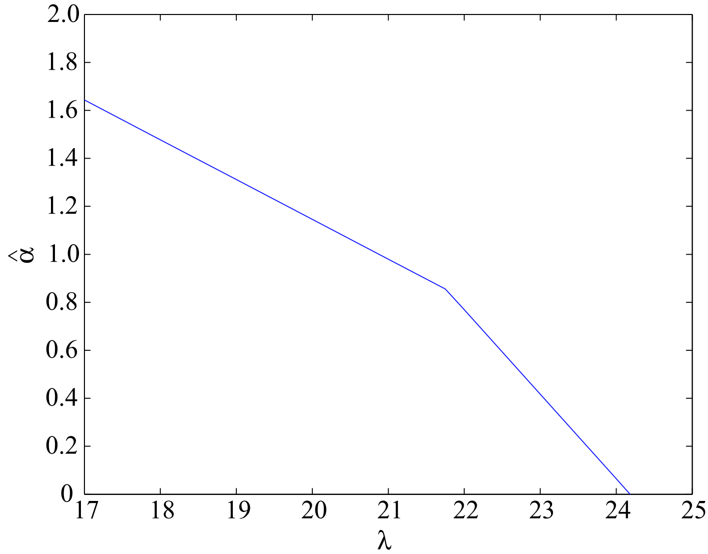

Figure 7. Variation of the robustness function $\hat{\alpha}$ of the 3×3 truss in Case 1 with respect to the performance requirement $\underline{\lambda}$.

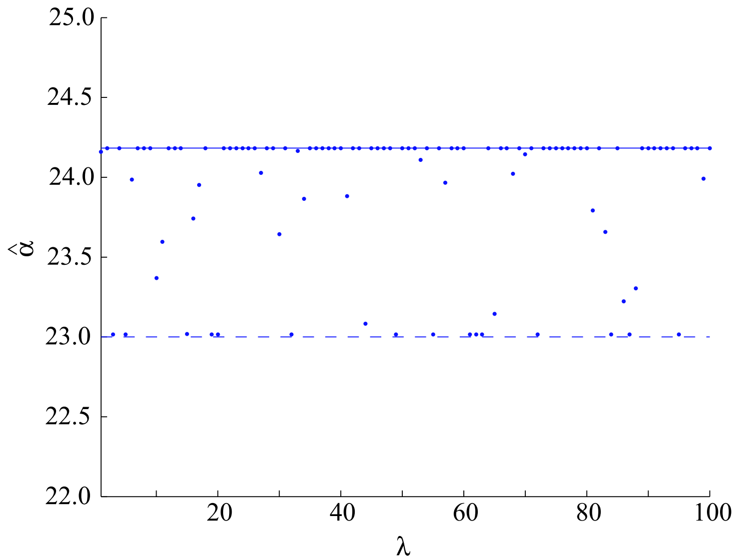

Figure 8. Limit load factor of the 3×3 truss in Case 4 for the uncertain $\boldsymbol{f}_{\rm R}$ corresponding to randomly generated $\boldsymbol{\zeta}$ (— nominal limit load factor $\lambda^*(\tilde{\boldsymbol{f}}_{\rm R}, \tilde{\boldsymbol{f}}_{\rm D})$; –– performance requirement $\underline{\lambda}$.

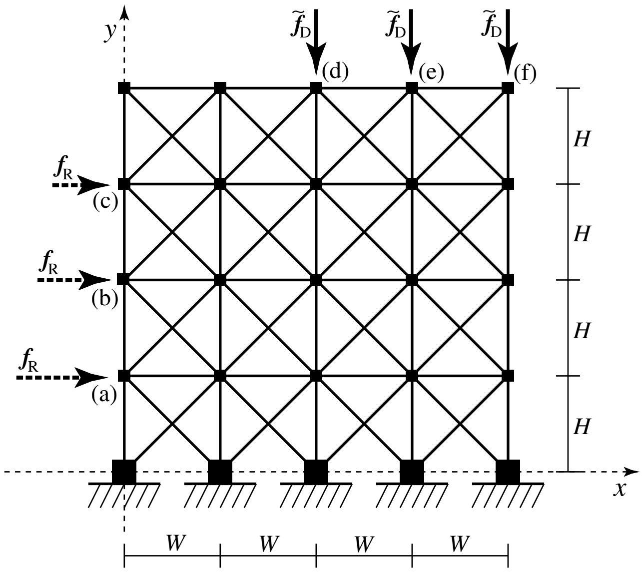

Figure 9. 68-member frame.

robustness functions are computed by solving the LP problems (28), (38), and (42) as $\hat{\alpha}(\underline{\lambda}) = 0.1739$, 0.1424, and 0.1207, respectively, in Case 3, Case 4, and Case 5. For Case 4, we randomly generate a number of $\boldsymbol{\zeta}$ satisfying $\|\boldsymbol{\zeta}\|_\infty = \hat{\alpha} = 0.1424$. The corresponding limit load factors $\lambda^*(\boldsymbol{f}_R(\boldsymbol{\zeta}), \tilde{\boldsymbol{f}}_D)$ are depicted in Figure 8. It is observed from Figure 8 that all generated limit load factors are not smaller than the performance requirement $\underline{\lambda}$. Moreover, there exists the case in which the limit load factor coincides with $\underline{\lambda}$.

5.2. ***68-member frame.*** Consider a plane frame illustrated in Figure 9, where $W = 200.0\,\text{cm}$ and $H = 200.0\,\text{cm}$. The intersecting pair of diagonals is not connected at their center. All lowest nodes are the fixed supports, that is, $n^{\text{d}} = 60$ and $n^{\text{m}} = 68$.

As the nominal dead load $\tilde{\boldsymbol{f}}_D$, we apply the external forces $(0, -300.0)\,\text{kN}$ at the nodes (d)–(f) as shown in Figure 9. The nominal reference disturbance load $\tilde{\boldsymbol{f}}_R$ is defined such that $(100.0, 0)\,\text{kN}$, $(70.0, 0)\,\text{kN}$, and $(40.0, 0)\,\text{kN}$ are applied at the nodes (a)–(c), respectively.

For each member, the yield criterion is defined by (46) with $\bar{q}_i = 1000.0\,\text{kN}$, $\bar{M}_i = 1000.0\,\text{kN}\cdot\text{m}$, and $N_i^{\text{p}} = 250.0\,\text{kN}$. The limit load factor under the nominal dead load is computed as $\lambda^*(\tilde{\boldsymbol{f}}_R, \tilde{\boldsymbol{f}}_D) = 28.58$ by employing the usual limit analysis, that is, by solving the LP problem (47).

Suppose that the dead load $\boldsymbol{f}_D$ has uncertainty and runs though the uncertainty set, Equation (10), while the reference disturbance load $\boldsymbol{f}_R$ is assumed to be certain. The uncertain dead load $F_0\,\boldsymbol{\zeta}$ is assumed to exist possibly at all free nodes except for the nodes (a)–(c). At the nodes (d)–(f), the uncertain dead load is supposed to exist in the direction of the x-axis. Note that the uncertain external moment is not considered, and hence $n^{\text{z}} = 31$ in Equation (10). The coefficient matrix F_0 is defined so that the uncertainties of components of $F_0\,\boldsymbol{\zeta}$ have no correlation, and each nonzero component of F_0 is equal to $100.0\,\text{kN}$.

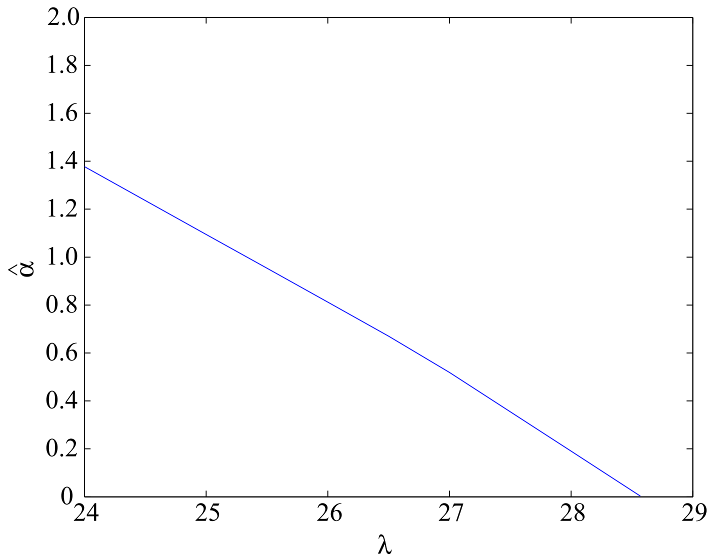

Figure 10. Variation of the robustness function $\hat{\alpha}$ of the 68-member frame with respect to the performance requirement $\underline{\lambda}$.

For a fixed $\underline{\lambda}$ and $p = 2$, the robustness function $\hat{\alpha}(\underline{\lambda})$ is computed by solving the LP problem (52). Figure 10 depicts the variation of the robustness function $\hat{\alpha}$ with respect to the performance requirement $\underline{\lambda}$. It is observed from Figure 10 that $\hat{\alpha}$ is a nonlinear function of $\underline{\lambda}$, since the collapse mode in the worst case depends on the magnitude of uncertainty of the dead load.

5.3. ***36-member frame.*** Consider a plane frame illustrated in Figure 11, where

$$W = 200.0\,\text{cm}, \qquad H = 200.0\,\text{cm},$$
$$n^{\text{d}} = 60, \qquad n^{\text{m}} = 36.$$

As the nominal dead load $\tilde{\boldsymbol{f}}_{\text{D}}$, we apply the external forces $(0, -300.0)$ kN at the nodes (d)–(h) as shown in Figure 11. The nominal reference disturbance load $\tilde{\boldsymbol{f}}_{\text{R}}$ is defined such that $(100.0, 0)$ kN, $(30.0, 0)$ kN, and $(20.0, 0)$ kN, respectively, are applied at the nodes (a), (c), and (d), respectively. The nominal limit load factor is computed as $\lambda^*(\tilde{\boldsymbol{f}}_{\text{R}}, \tilde{\boldsymbol{f}}_{\text{D}}) = 15.00$.

Suppose that the dead load $\boldsymbol{f}_{\text{D}}$ has uncertainty such that $\boldsymbol{f}_{\text{D}} \in \mathscr{D}_2(\alpha, \tilde{\boldsymbol{f}}_{\text{D}})$, while the reference disturbance load $\boldsymbol{f}_{\text{R}}$ is assumed to be certain. The uncertain dead load $F_0\,\boldsymbol{\zeta}$ is assumed to exist possibly at all free nodes except for the nodes (a)–(d). At the nodes (e)–(h), the uncertain dead forces are supposed to exist in the direction of the x-axis. Note that the uncertain external moment is not considered, and hence $n^{\text{z}} = 28$ in Equation (10). The coefficient matrix F_0 is defined so that the uncertainties of components of $F_0\,\boldsymbol{\zeta}$ have no correlation, and each nonzero component of F_0 is equal to 100.0 kN. Figure 12 depicts the variation of the robustness function $\hat{\alpha}$ with respect to the performance requirement $\underline{\lambda}$. It is observed from Figure 12 that $\hat{\alpha}$ is a nonlinear function of $\underline{\lambda}$.

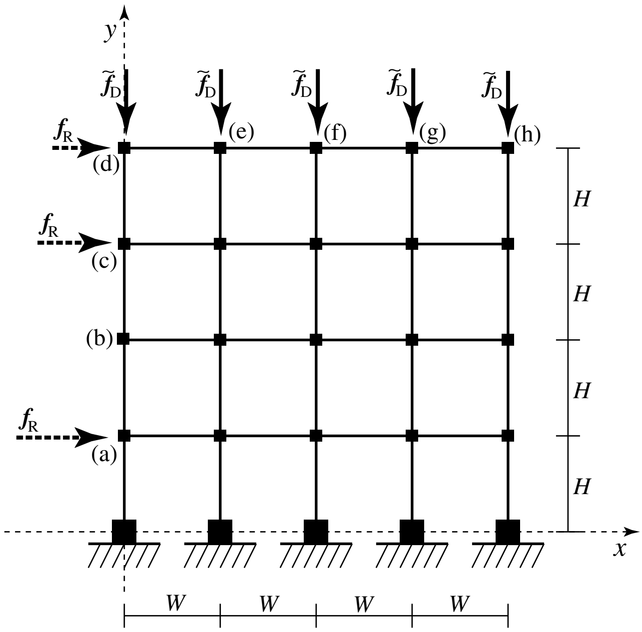

Figure 11. 36-member frame.

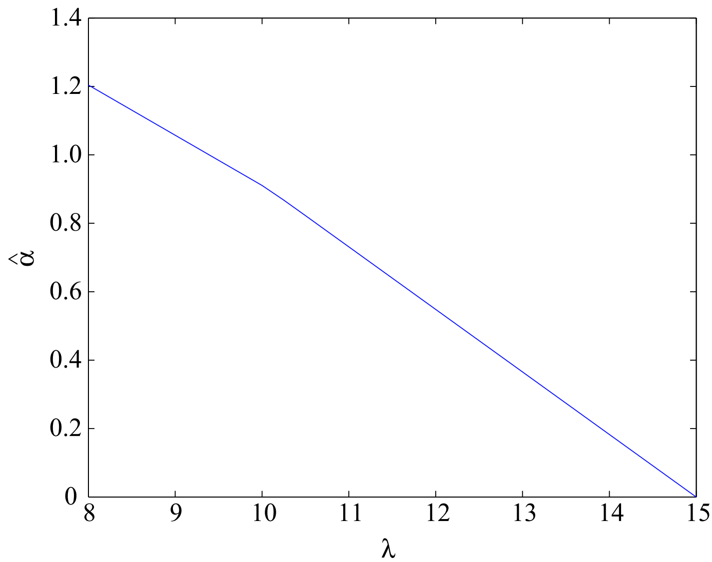

Figure 12. Variation of the robustness function $\hat{\alpha}$ of the 36-member frame with respect to the performance requirement $\underline{\lambda}$.

6. Robust optimization and level of uncertainty

The robust optimization problem is investigated under the limit load factor constraint with the uncertain dead load. The level α of the uncertainty is fixed throughout this section.

6.1. ***Robust optimization of trusses.*** In this section, we formulate the robust optimization problem of trusses associated with the limit load factor by utilizing the theoretical results in Section 3. The notation introduced in Section 3 are used in this section again.

Let l_i denote the length of the ith member. The vector of member cross-sectional areas is denoted by $\boldsymbol{a} = (a_i) \in \Re^{n^{\mathrm{m}}}$, which is regarded as a design variable vector. Without uncertainty of the external load, we first consider the minimization problem of the structural volume over the lower bound constraint of the limit load factor, which is formulated as

$$\min_{\boldsymbol{a}} \boldsymbol{l}^\top \boldsymbol{a} \quad \text{such that} \quad \begin{cases} \lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}}) \geq \underline{\lambda}, \\ \boldsymbol{a} \geq \boldsymbol{0}, \end{cases} \tag{57}$$

Note that the limit load factor $\lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}})$ depends on $\boldsymbol{a}$ implicitly in (57), because the absolute value of admissible axial force $\bar{q}_i$ in (8) depends on a_i as (7). We can regard (57) as the nominal (or conventional) structural optimization problem associated with the limit load factor.

Suppose that $\boldsymbol{f}_{\mathrm{D}}$ is uncertain obeying the uncertainty model (10) for the fixed $\alpha \in \Re_{++}$, while $\boldsymbol{f}_{\mathrm{R}}$ is assumed to be certain. For simplicity, we write $\boldsymbol{f}_{\mathrm{R}} = \tilde{\boldsymbol{f}}_{\mathrm{R}}$ in the remainder of this section. The robust counterpart problem [Ben-Tal and Nemirovski 2002] of (57) is formulated as

$$\min_{\boldsymbol{a}} \boldsymbol{l}^\top \boldsymbol{a} \quad \text{such that} \quad \begin{cases} \lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}}) \geq \underline{\lambda} \quad (\text{for all } \boldsymbol{f}_{\mathrm{D}} \in \mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{D}})), \\ \boldsymbol{a} \geq \boldsymbol{0}. \end{cases} \tag{58}$$

Note again that α is fixed in Equation (58), while α has been regarded as a variable in (11). Observe that the constraint

$$\lambda^*(\boldsymbol{f}_{\mathrm{R}}, \boldsymbol{f}_{\mathrm{D}}) \geq \underline{\lambda} \quad (\text{for all } \boldsymbol{f}_{\mathrm{D}} \in \mathcal{D}_p(\alpha, \tilde{\boldsymbol{f}}_{\mathrm{D}})) \tag{59}$$

becomes active at an optimal solution of (58). Hence, the robustness function of the optimal solution of (58) is given by

$$\hat{\alpha}(\underline{\lambda}) = \alpha. \tag{60}$$

It follows from the result of Proposition 3.2 that the robust constraint (59) of (58) is equivalently rewritten as

$$\begin{aligned} \boldsymbol{h}_i^\dagger(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}} + \tilde{\boldsymbol{f}}_{\mathrm{D}}) + \alpha \|\boldsymbol{h}_i^\dagger F_0\|_{p^*} + \boldsymbol{h}_i^\perp \boldsymbol{\xi} \leq \bar{q}_i, \quad i = 1, \ldots, n^{\mathrm{m}}, \\ -\boldsymbol{h}_i^\dagger(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}} + \tilde{\boldsymbol{f}}_{\mathrm{D}}) + \alpha \|\boldsymbol{h}_i^\dagger F_i\|_{p^*} - \boldsymbol{h}_i^\perp \boldsymbol{\xi} \leq \bar{q}_i, \quad i = 1, \ldots, n^{\mathrm{m}}. \end{aligned}$$

Consequently, by using Equation (7), the problem (58) is equivalent to the following LP problem in the variables $\boldsymbol{a}$ and $\boldsymbol{\xi}$:

$$\min_{\boldsymbol{a},\boldsymbol{\xi}} \boldsymbol{l}^\top \boldsymbol{a} \text{ such that } \begin{cases} \boldsymbol{h}_i^\dagger(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}} + \tilde{\boldsymbol{f}}_{\mathrm{D}}) + \alpha\|\boldsymbol{h}_i^\dagger F_0\|_{p^*} + \boldsymbol{h}_i^\perp\boldsymbol{\xi} \le \bar{\sigma}_i a_i, & i = 1,\dots,n^{\mathrm{m}}, \\ -\boldsymbol{h}_i^\dagger(\underline{\lambda}\boldsymbol{f}_{\mathrm{R}} + \tilde{\boldsymbol{f}}_{\mathrm{D}}) + \alpha\|\boldsymbol{h}_i^\dagger F_i\|_{p^*} - \boldsymbol{h}_i^\perp\boldsymbol{\xi} \le \bar{\sigma}_i a_i, & i = 1,\dots,n^{\mathrm{m}}, \\ \boldsymbol{a} \ge \boldsymbol{0}. \end{cases} \tag{61}$$

It is rather amazing that the robust optimization problem (58) can be reformulated into the LP problem (61).

Similarly, it can be shown that the robust optimization problem under the uncertain reference disturbance load $\boldsymbol{f}_{\mathrm{R}}$ is also reformulated into an LP problem, if $\boldsymbol{f}_{\mathrm{R}}$ obeys the uncertainty model introduced in Section 3.3.

6.2. ***Robust optimization of framed structures.*** In this section, we show that a robust optimization problem of frames associated with the limit load factor can be reformulated as an LP problem. The notation introduced in Section 4 are used in this section again.

Let l_i and a_i denote the length and cross-sectional area of the ith member, respectively. Consider the sandwich cross-section with the radius d_i, the moment of inertia of which is written as $t_i = d_i^2 a_i$. Then $\bar{q}_i$ and $\bar{m}_i$ in (46) are written as

$$\bar{q}_i = \bar{\sigma}_i a_i, \quad \bar{m}_i = \bar{\sigma}_i d_i a_i. \tag{62}$$

Provided that d_i is fixed, we can assume that only $\boldsymbol{a}$ is the design variables vector. Hence, the nominal optimization problem of frames can be formulated in the form of (57).

Suppose that $\boldsymbol{f}_{\mathrm{D}}$ is uncertain and obeys the uncertainty model (10) for the fixed $\alpha \in \Re_{++}$. The robust counterpart of the optimization problem is formulated in the form of (58). In a manner similar to Section 6.1, it follows from the result of Proposition 4.1 that the robust constraint (59) for frames is equivalently rewritten into the constraints of (52). Consequently, by using (62), the problem (58) for frames is equivalent to the following LP problem in the variables $\boldsymbol{a}$ and $\boldsymbol{\xi}$:

$$\begin{aligned} &\min_{\boldsymbol{a},\boldsymbol{\xi}} \boldsymbol{l}^\top \boldsymbol{a} \text{ such that} \\ &\quad \left[(-1)^\mu \boldsymbol{h}_i^q + (-1)^\nu \boldsymbol{h}_{i,j}^m/d_i\right](\underline{\lambda}\boldsymbol{f}_{\mathrm{R}}^0 + \tilde{\boldsymbol{f}}_{\mathrm{D}}^0) + (-1)^\mu q_i^{\mathrm{p}} + \alpha \left\| \left[(-1)^\mu \boldsymbol{h}_i^q + (-1)^\nu \boldsymbol{h}_{i,j}^m/d_i\right] F_0 \right\|_{p^*} \\ &\qquad + \left[(-1)^\mu \bar{\boldsymbol{h}}_i^q + (-1)^\nu \bar{\boldsymbol{h}}_{i,j}^m/d_i\right]\boldsymbol{\xi} \le \bar{\sigma}_i a_i, \quad i = 1,\dots,n^{\mathrm{m}}, (j,\mu,\nu) \in \{1,2\}^3, \quad \boldsymbol{a} \ge \boldsymbol{0}. \end{aligned} \tag{63}$$

6.3. ***Level of uncertainty and optimal structural volume.*** By using the LP formulations (61) and (62), we investigate the relation between the level of uncertainty α and the structural volume $\boldsymbol{l}^\top \boldsymbol{a}$ at the optimal design of the robust optimization problem.

6.3.1. *Truss example.* Recall the 3×3 truss illustrated in Figure 1, the loading condition of which has been defined in Section 5.1. Consider the robust optimization problem (58) with $\underline{\lambda} = 23.0$ and $\alpha = 0.4$. The robust optimal design found by solving (61) is shown in Figure 13, where the width of each member

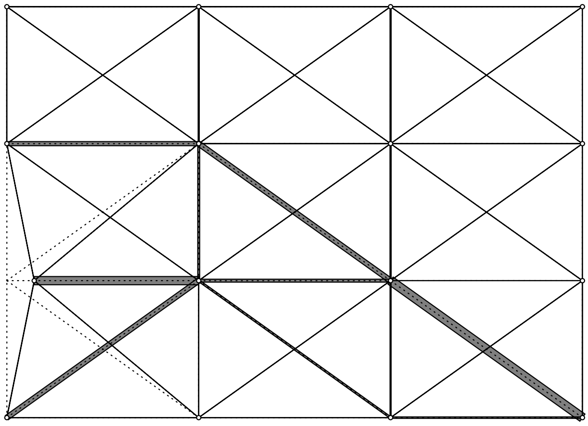

Figure 13. Robust optimal design of the 3 × 3 truss at $(\hat{\alpha}, \underline{\lambda}) = (0.4, 23.0)$ and the collapse mode with the nominal external load.

is proportional to its cross-sectional area. The limit load factor of this design under the nominal dead load is $\lambda^*(\tilde{\boldsymbol{f}}_{\rm R}, \tilde{\boldsymbol{f}}_{\rm D}) = 23.57$. The corresponding collapse mode is also illustrated in Figure 13.

Figure 14 depicts the relation between the structural volume and the robustness function at the optimal design. Note again that (60) holds at an optimal solution of (61). Moreover, the optimal solution of (61) at $\alpha = 0$ coincides with the optimal solution of the nominal optimization problem (57). It is of interest to note that, from the definition of the robustness function, any truss design satisfying the constraint (59) with $\underline{\lambda} = 24.0$ is plotted in (or on the boundary of) the domain $\mathscr{F}$ in Figure 14. Thus, engineers may be able to make decisions incorporating the tradeoff between the robustness and the structural volume by using Figure 14. Note that the optimal value of the problem (61) depends linearly on α if the active set of constraints does not change when α increases. Hence, the optimal structural volume is a piecewise linear function of α in this example.

6.3.2. *Frame example.* Recall the 68-member frame illustrated in Figure 9, the loading condition of which has been defined in Section 5.2. The cross-section of each member is assumed to be sandwich, where $d_i = 1.0$ for simplicity. Supposing that $\boldsymbol{f}_{\rm D}$ is uncertain, consider the robust optimization problem (58), where $\underline{\lambda} = 27.0$ and $\alpha = 0.5$.

The robust optimal design found by solving Equation (62) is shown in Figure 15, where the width of each member is proportional to its cross-sectional area. The limit load factor of this design under the nominal dead load is $\lambda^*(\tilde{\boldsymbol{f}}_{\rm R}, \tilde{\boldsymbol{f}}_{\rm D}) = 28.47$. Figure 16 depicts the relation between the optimal structural volume and the robustness function for various values of $\underline{\lambda}$.

7. Conclusions

In this paper, we have proposed tractable numerical methods for robustness analysis of structures associated with the limit load factor under the load uncertainties. Particularly, it has been shown that the info-gap robustness function can be obtained by solving a linear programming (LP) problem. The effective method for computing the robustness function may permit us to apply the info-gap decision theory

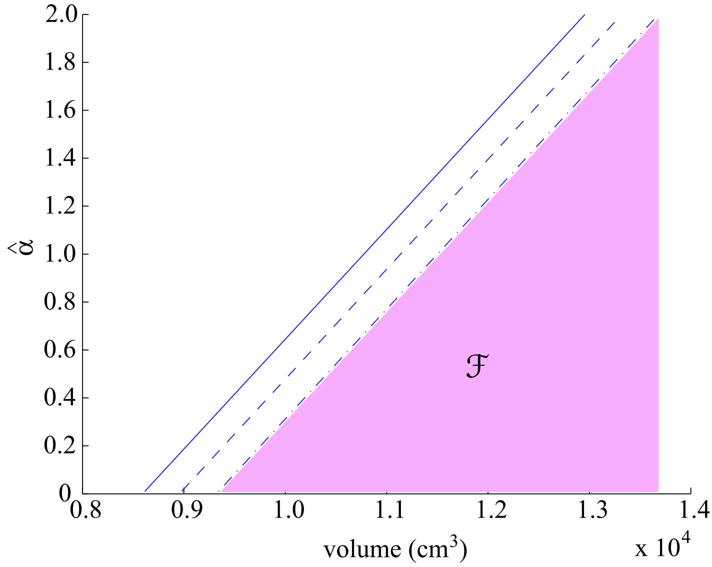

Figure 14. Relation between the robustness function $\hat{\alpha}$ and the optimal structural volume of the 3 × 3 truss (— $\underline{\lambda} = 22.0$; −− $\underline{\lambda} = 23.0$; −·− $\underline{\lambda} = 24.0$).

[Ben-Haim 2006] to designing structures which never encounter violation of mechanical performance constraints under the uncertainty considered.

A main contribution of this paper is to show that the robustness function associated with the constraint on the limit load factor can be obtained as the optimal value of an LP problem. It is rather amazing that the robustness function can be computed easily by solving an LP problem, because the robustness function is originally defined in terms of the optimization problem over the infinitely many constraints. Moreover, for the given magnitude of uncertainty, detecting the worst-case limit load factor corresponds to finding a

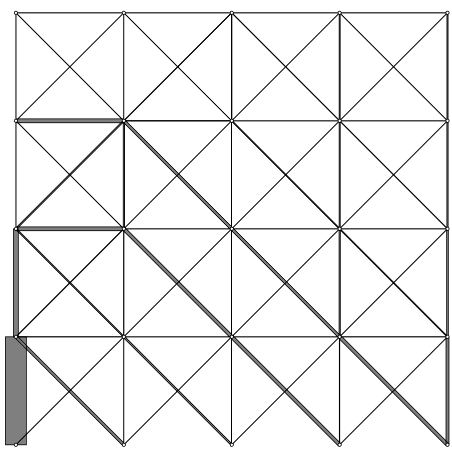

Figure 15. Robust optimal solution of the 68-member frame at $(\hat{\alpha}, \underline{\lambda}) = (0.5, 27.0)$.

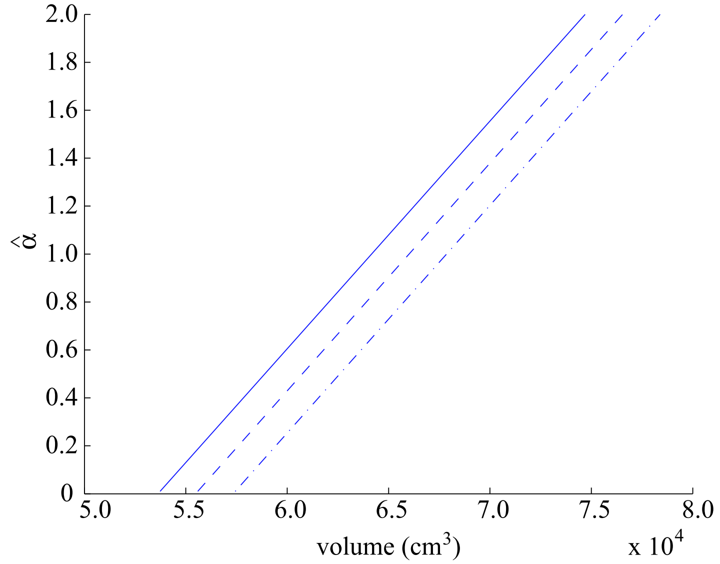

Figure 16. Relation between the robustness function $\hat{\alpha}$ and the optimal structural volume of the 68-member frame (— $\underline{\lambda} = 26.0$; −− $\underline{\lambda} = 27.0$; −·− $\underline{\lambda} = 28.0$).

global optimal solution of a nonlinear optimization problem as discussed in [Kanno and Takewaki 2007], that is, the results of this paper imply that computation of the robustness function is much easier than finding the worst case. Thus, we have shown that the constraint on the limit load factor is regarded as a tractable class of problems for computing the robustness function, although it is very difficult to compute the exact value of the robustness function in general.

We can compare the robustness of structures quantitatively by using the robustness function. In the numerical examples, the robustness function has been computed for uncertain trusses and framed structures by solving LP problems. The nonlinear relation between the robustness function and the performance requirement has been observed. It should be emphasized that most convex model approaches for robustness and/or uncertainty analysis have been developed based on first-order perturbation, while the proposed method does not use any approximation. Hence, the method presented is valid even for a large magnitude of uncertainty.

As a second contribution, the robust structural optimization associated with the limit load factor has been formulated for a given magnitude of uncertainty. It has been shown that this robust optimization problem can be reformulated as an LP problem for trusses as well as frames with sandwich cross-sections. In the numerical examples, robust optimal designs of a truss and frame are computed by solving LP problems. The relation between the robustness function and the optimal structural volume has been investigated by solving the robust optimization problems for various magnitudes of uncertainty.

Acknowledgment

The authors are grateful to Izuru Takewaki for his helpful comments. They are also grateful to anonymous referees for valuable comments and suggestions.

References

[Alefeld and Mayer 2000] G. Alefeld and G. Mayer, "Interval analysis: theory and applications", *J. Comput. Appl. Math.* **121** (2000), 421–464.

[Andersen et al. 1998] K. Andersen, E. Christiansen, and M. L. Overton, "Computing limit loads by minimizing a sum of norms", *SIAM J. Sci. Comput.* **19** (1998), 1046–1062.

[Ben-Haim 2006] Y. Ben-Haim, *Information-gap decision theory: decisions under severe uncertainty*, 2nd ed., Academic Press, London, UK, 2006.

[Ben-Haim and Elishakoff 1990] Y. Ben-Haim and I. Elishakoff, *Convex models of uncertainty in applied mechanics*, Elsevier, Amsterdam, The Netherlands, 1990.

[Ben-Tal and Nemirovski 1997] A. Ben-Tal and A. Nemirovski, "Robust truss topology optimization via semidefinite programming", *SIAM J. Optimiz.* **7** (1997), 991–1016.

[Ben-Tal and Nemirovski 2002] A. Ben-Tal and A. Nemirovski, "Robust optimization—methodology and applications", *Math. Program.* **B92** (2002), 453–480.

[Beyer and Sendhoff 2007] H.-G. Beyer and S. Sendhoff, "Robust optimization—a comprehensive survey", *Comput. Methods Appl. Mech. Eng.* **196** (2007), 3190–3218.

[Chen et al. 2002] S. Chen, H. Lian, and X. Yang, "Interval static displacement analysis for structures with interval parameters", *Int. J. Numer. Methods Eng.* **53** (2002), 393–407.

[Cocchetti and Maier 2003] G. Cocchetti and G. Maier, "Elastic-plastic and limit-state analyses of frames with softening plastic-hinge models by mathematical programming", *Int. J. Solids Struct.* **40** (2003), 7219–7244.

[Elishakoff et al. 1994] I. Elishakoff, R. T. Haftka, and J. Fang, "Structural design under bounded uncertainty—optimization with anti-optimization", *Comput. Struct.* **53** (1994), 1401–1405.

[Ganzerli and Pantelides 1999] S. Ganzerli and C. P. Pantelides, "Load and resistance convex models for optimum design", *Struct. Optimization* **17** (1999), 259–268.

[Hodge 1959] P. G. Hodge, *Plastic analysis of structures*, McGraw-Hill, New York, 1959.

[Kanno and Takewaki 2006a] Y. Kanno and I. Takewaki, "Robustness analysis of trusses with separable load and structural uncertainties", *Int. J. Solids Struct.* **43** (2006), 2646–2669.

[Kanno and Takewaki 2006b] Y. Kanno and I. Takewaki, "Sequential semidefinite program for robust truss optimization based on robustness functions associated with stress constraints", *J. Optim. Theory Appl.* **130** (2006), 265–287.

[Kanno and Takewaki 2007] Y. Kanno and I. Takewaki, "Worst-case plastic limit analysis of trusses under uncertain loads via mixed 0-1 programming", *J. Mech. Mater. Struct.* **2** (2007), 245–273.

[Kharmanda et al. 2004] G. Kharmanda, N. Olhoff, A. Mohamed, and M. Lemaire, "Reliability-based topology optimization", *Struct. Multidiscip. Optimz.* **26** (2004), 295–307.

[Krabbenhoft and Damkilde 2003] K. Krabbenhoft and L. Damkilde, "A general non-linear optimization algorithm for lower bound limit analysis", *Int. J. Numer. Methods Eng.* **56** (2003), 165–184.

[Llyoyd Smith et al. 1990] D. Llyoyd Smith, P.-H. Chuang, and J. Munro, "Fuzzy linear programming in plastic limit design", pp. 425–435 in *Mathematical Programming Methods in Structural Plasticity*, edited by D. Lloyd Smith, Springer-Verlag, Wien, Austria, 1990.

[Marti and Stoeckel 2004] K. Marti and G. Stoeckel, "Stochastic linear programming methods in limit load analysis and optimal plastic design under stochastic uncertainty", *ZAMM* **10** (2004), 666–677.

[MatLab 2006] The MathWorks, *Using MATLAB*, Natick, MA: The MathWorks, 2006.

[Michael Sttele 2004] J. Michael Sttele, *The Cauchy-Schwarz master class*, Cambridge University Press, New York, NY, 2004.

[Muhanna and Mullen 2001] R. L. Muhanna and R. L. Mullen, "Uncertainty in mechanics problems—interval-based approach", *J. Eng. Mech. (ASCE)* **127** (2001), 557–566.

[Muralidhar and Jagannatha Rao 1997] R. Muralidhar and J. R. Jagannatha Rao, "New models for optimal truss topology in limit design based on unified elastic/plastic analysis", *Comput. Methods Appl. Mech. Eng.* **140** (1997), 109–138.

[Qiu and Elishakoff 1998] Z. Qiu and I. Elishakoff, "Antioptimization of structures with large uncertain-but-non-random parameters via interval analysis", *Comput. Methods Appl. Mech. Eng.* **152** (1998), 361–372.

[Rocho and Sonnenberg 2003] P. Rocho and S. Sonnenberg, "Limit analysis of frames—application to structural reliability", pp. 269–282 in *Numerical methods for limit and shakedown analysis* (John von Neumann Institute for Computing), edited by M. Staat and M. Heitzer, Jülich, Germany, 2003.

[Staat and Heitzer 2003] M. Staat and M. Heitzer, "Probabilistic limit and shakedown problems", pp. 217–268 in *Numerical methods for limit and shakedown analysis* (John von Neumann Institute for Computing), edited by M. Staat and M. Heitzer, Jülich, Germany, 2003.

[Takewaki and Ben-Haim 2005] I. Takewaki and Y. Ben-Haim, "Info-gap robust design with load and model uncertainties", *J. Sound Vib.* **288** (2005), 551–570.

[Zang et al. 2005] C. Zang, M. I. Friswell, and J. E. Mottershead, "A review of robust optimal design and its application in dynamics", *Comput. Struct.* **83** (2005), 315–326.

Received 25 Jun 2007. Revised 7 Aug 2007. Accepted 8 Aug 2007.

YU MATSUDA: y-matsuda@ipl.t.u-tokyo.ac.jp
Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan

YOSHIHIRO KANNO: kanno@mist.i.u-tokyo.ac.jp
Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan
http://www.simplex.t.u-tokyo.ac.jp/