Worst Scenario of Deficiency of Structural Elements in Plastic Limit Analysis

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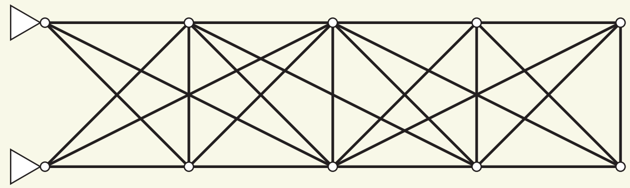
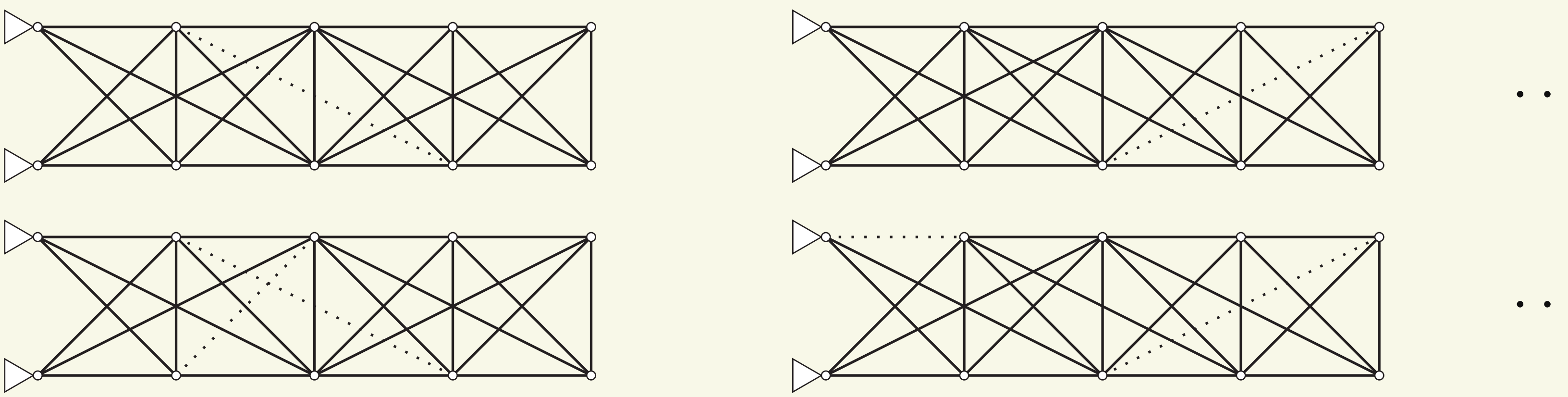
May 23, 2012

worst damage scenario

- nominal (undamaged) structure:



- damage scenarios



- Which is the worst?

  —the severest degradation of structural performance

motivation: measure of structural redundancy

- degree of static determinacy $s = n - \mathrm{rank}\, H$

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- strength redundancy factor $r = l_{\mathrm{intact}}/(l_{\mathrm{intact}} - l_{\mathrm{damaged}})$

  [Frangopol & Curley 87]

  - l_{intact} : ultimate strength of the intact structure
  - l_{damaged} : ultimate strength of the damaged structure

motivation: measure of structural redundancy

- degree of static determinacy $s = n - \operatorname{rank} H$
- strength redundancy factor $r = l_{\rm intact}/(l_{\rm intact} - l_{\rm damaged})$

  [Frangopol & Curley 87]

  - $l_{\rm intact}$: ultimate strength of the intact structure
  - $l_{\rm damaged}$: ultimate strength of the damaged structure

- sensitivity index $1/r$ [Ohi, Ito & Li 04]
  - “ultimate strength” $=$ limit load factor
  - “damage” $=$ loss of a member

motivation: measure of structural redundancy

- degree of static determinacy $s = n - \mathrm{rank}\, H$
- strength redundancy factor $r = l_{\mathrm{intact}}/(l_{\mathrm{intact}} - l_{\mathrm{damaged}})$ [Frangopol & Curley 87]
- sensitivity index $1/r$ [Ohi, Ito & Li 04]
- $(P(D) - P(C))/P(C)$ [Fu & Frangopol 90]
  - $P(C)$: pr. of system collapse
  - $P(D)$: pr. of failure of a structural component

  or [Hendawi & Frangopol 94]
  - $P(C)$: pr. of collapse
  - $P(D)$: pr. that any first-member-yielding occurs

motivation: measure of structural redundancy

- degree of static determinacy $s = n - \operatorname{rank} H$
- strength redundancy factor $r = l_{\rm intact}/(l_{\rm intact} - l_{\rm damaged})$
  [Frangopol & Curley 87]
- sensitivity index $1/r$ [Ohi, Ito & Li 04]
- $(P(D) - P(C))/P(C)$ [Fu & Frangopol 90]
- residual strength index $l_i/l_{\rm u}$ [Feng & Moses 86]
  - $l_{\rm u}$: ultimate strength
  - l_i : strength after the ith structural component has failed

motivation: measure of structural redundancy

- degree of static determinacy $s = n - \mathrm{rank}\, H$
- strength redundancy factor $r = l_{\mathrm{intact}}/(l_{\mathrm{intact}} - l_{\mathrm{damaged}})$
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- residual strength index l_i/l_{u} [Feng & Moses 86]
- redundancy-strength index $l_{\mathrm{u}}/l_{\mathrm{y}}$ [Husain & Tsopelas 04]
  - l_{u} : ultimate strength
  - l_{y} : strength at “the first significant yielding”

motivation: measure of structural redundancy

- degree of static determinacy $s = n - \mathrm{rank}\, H$
- strength redundancy factor $r = l_{\mathrm{intact}}/(l_{\mathrm{intact}} - l_{\mathrm{damaged}})$
  [Frangopol & Curley 87]
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- redundancy-strength index $l_{\mathrm{u}}/l_{\mathrm{y}}$ [Husain & Tsopelas 04]
- strong redundancy [Kanno & Ben-Haim 11]
  - greatest level of deficiency
    without violating the performance requirement

motivation: measure of structural redundancy

- degree of static determinacy $s = n - \operatorname{rank} H$
- strength redundancy factor $r = l_{\rm intact}/(l_{\rm intact} - l_{\rm damaged})$ [Frangopol & Curley 87]
- sensitivity index $1/r$ [Ohi, Ito & Li 04]
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- strong redundancy [Kanno & Ben-Haim 11]

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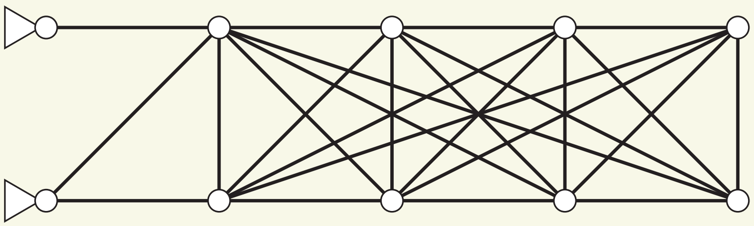
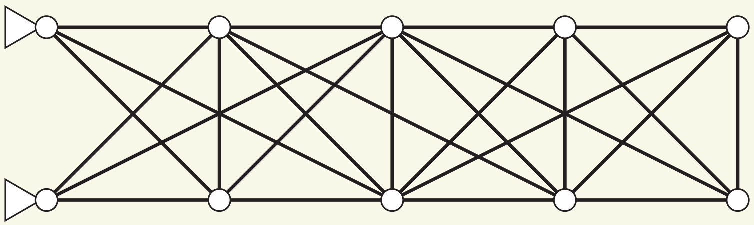
high redundancy—small degradation of performance
when some structural components fail
$\rightarrow$ attempt to find worst failure scenario

importance of w. s. in assessing redundancy/robustness

- strong redundancy [Kanno & Ben-Haim 11]
- Which truss has higher redundancy?



(A)

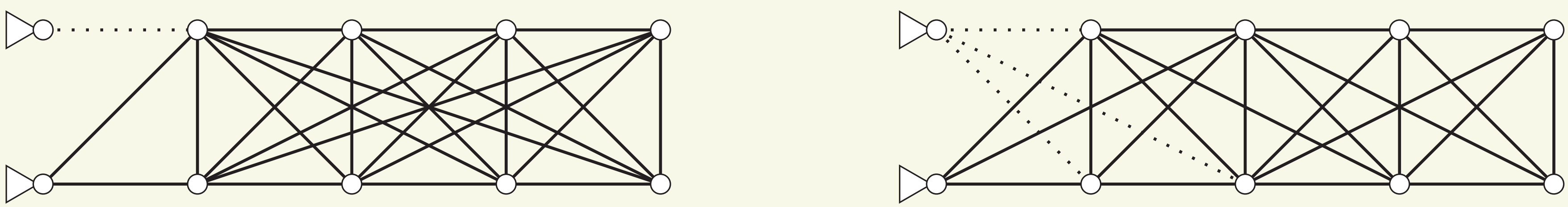


(B)

  - # of members $= 25$
  - deg. of static indeterminacy $= 9$

importance of w. s. in assessing redundancy/robustness

- strong redundancy [Kanno & Ben-Haim 11]
- Which truss has higher redundancy?
  - Concerning the stability constraint, (A) $<$ (B), because



(A′)
strong redundancy $= 0$

(B′)
strong redundancy $= 2$

importance of w. s. in assessing redundancy/robustness

- strong redundancy [Kanno & Ben-Haim 11]
- Which truss has higher redundancy?
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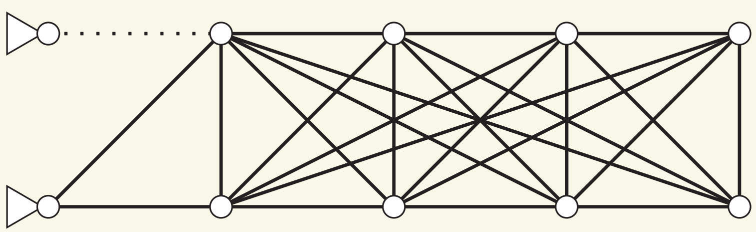
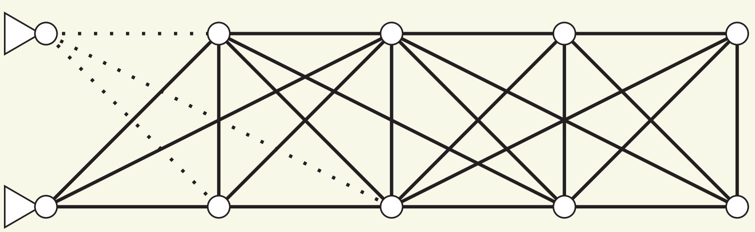


(A′)
strong redundancy $= 0$

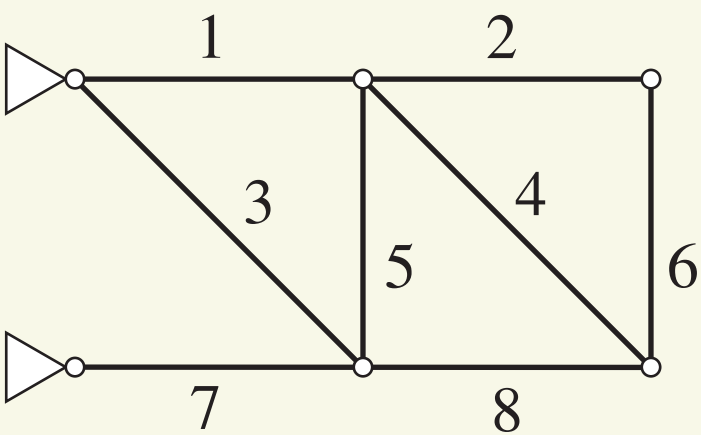


(B′)
strong redundancy $= 2$

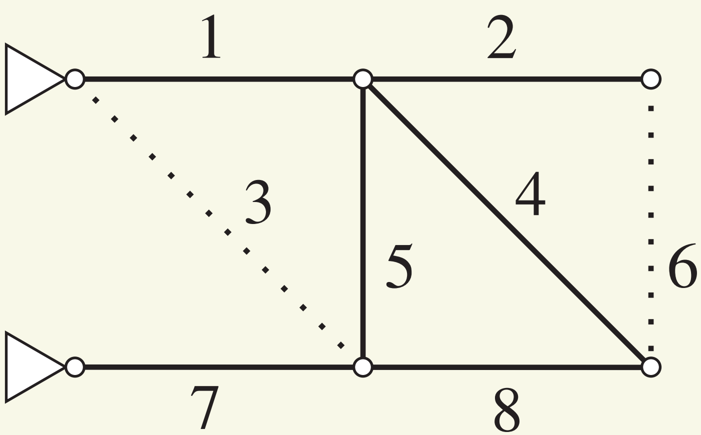
  - (A′) and (B′) are worst scenarios.
  - Redundancy might be assessed
    in the worst deficiency of structural components.
  - Def. of “worst” depends on performance requirement.

deficiency of members

- $t_i \in \{0, 1\}$: indicator of soundness of member i
  - $\tilde{\boldsymbol{t}} = (1, 1, \dots, 1)$: nominal scenario (no damage)
  - $t_i = \begin{cases} 1 & \text{member } i \text{ is present} \\ 0 & \text{member } i \text{ is absent} \end{cases}$



$\boldsymbol{t} = (1, 1, 1, 1, 1, 1, 1, 1)$



$\boldsymbol{t} = (1, 1, 0, 1, 1, 0, 1, 1)$

deficiency of members

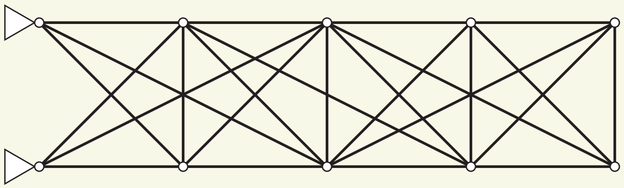
- $t_i \in \{0, 1\}$: indicator of soundness of member i
  - $\tilde{\boldsymbol{t}} = (1, 1, \dots, 1)$: nominal scenario (no damage)
  - $t_i = \begin{cases} 1 & \text{member } i \text{ is present} \\ 0 & \text{member } i \text{ is absent} \end{cases}$
- assumptions:
  - At most α members are missing from $\tilde{\boldsymbol{t}}$, due to damage, failure, aging, or fire, etc.
  - We do not know in advance which members are missing.

introduced by [Kanno & Ben-Haim 11]

uncertainty model of structural deficiency

$$\mathcal{T}(\alpha; \tilde{\boldsymbol{t}}) = \left\{ \boldsymbol{t} \in \{0,1\}^m \mid \boldsymbol{t} \leq \tilde{\boldsymbol{t}},\ \sum_{i=1}^{m} |\tilde{t}_i - t_i| \leq \alpha \right\}$$

- ex.) $\alpha = 1$



$\tilde{t}$ (nominal)

- $\mathcal{T}(0; \tilde{\boldsymbol{t}}) = \{\tilde{\boldsymbol{t}}\}$
- $\alpha < \alpha' \ \Rightarrow\ \mathcal{T}(\alpha; \tilde{\boldsymbol{t}}) \subseteq \mathcal{T}(\alpha'; \tilde{\boldsymbol{t}})$

uncertainty model of structural deficiency

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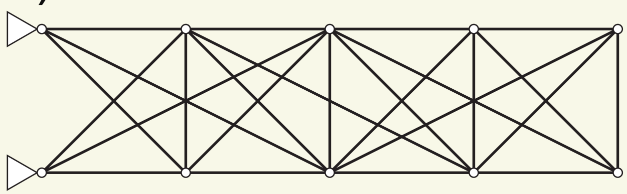
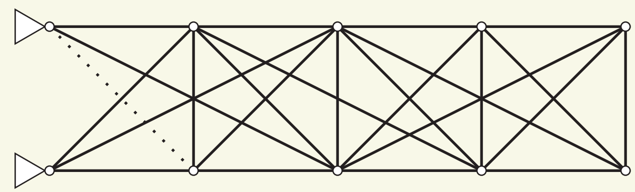
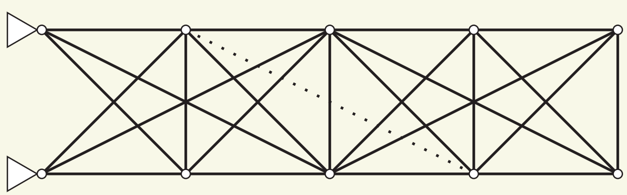


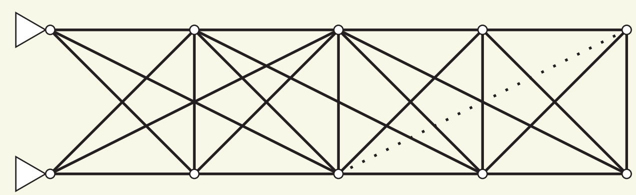
$\tilde{\boldsymbol{t}}$ (nominal)



$\tilde{\boldsymbol{t}} - 1$ member



$\tilde{\boldsymbol{t}} - 1$ member



$\tilde{\boldsymbol{t}} - 1$ member

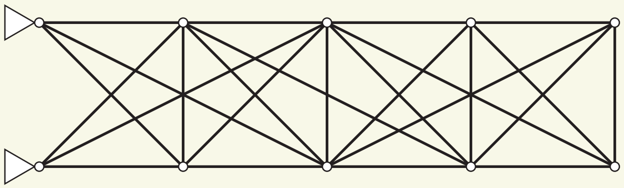
$\cdots$

- α : level of uncertainty

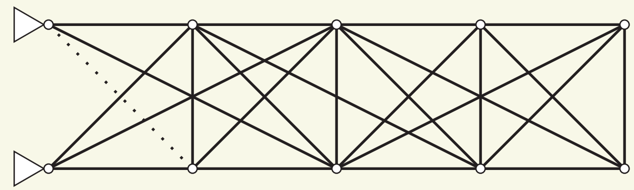
uncertainty model of structural deficiency

$$\mathcal{T}(\alpha;\tilde{\boldsymbol{t}}) = \left\{ \boldsymbol{t} \in \{0,1\}^m \mid \boldsymbol{t} \le \tilde{\boldsymbol{t}},\ \sum_{i=1}^{m} |\tilde{t}_i - t_i| \le \alpha \right\}$$

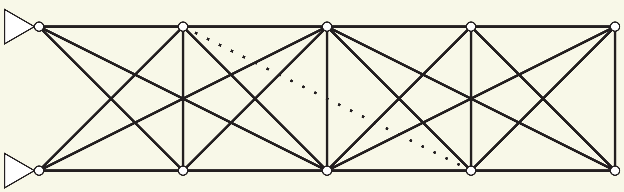
- ex.) $\alpha = 2$



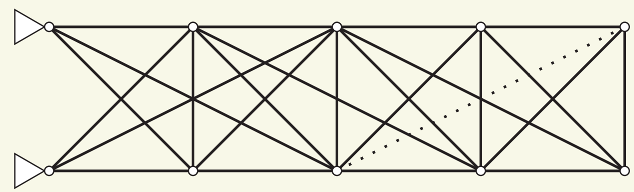
$\tilde{\boldsymbol{t}}$ (nominal)



$\tilde{\boldsymbol{t}} - 1$ member

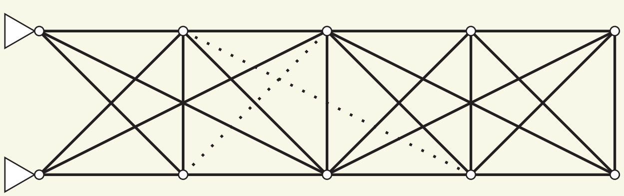


$\tilde{\boldsymbol{t}} - 1$ member

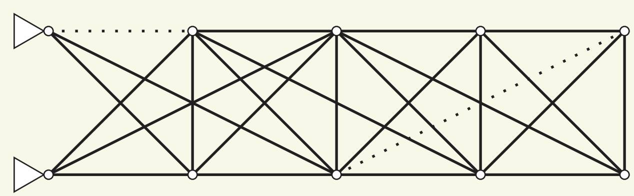


$\tilde{\boldsymbol{t}} - 1$ member

$\cdots$



$\tilde{\boldsymbol{t}} - 2$ members



$\tilde{\boldsymbol{t}} - 2$ members

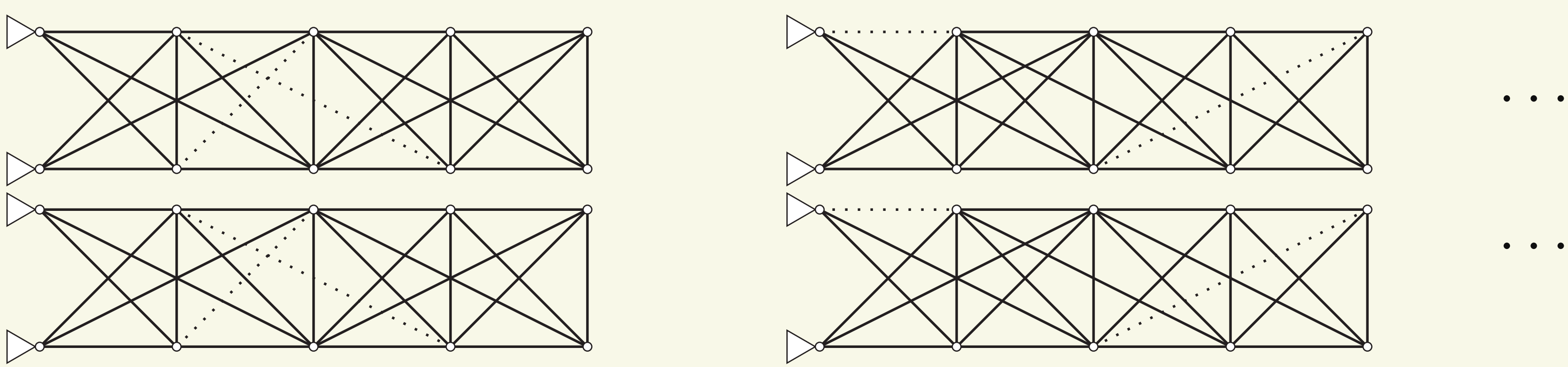
$\cdots$

worst scenario problem

- $\mathcal{T}(\alpha;\tilde{\boldsymbol{t}})$: set of deficiency scenarios
- $\lambda^*(\boldsymbol{t})$: limit load factor $\leftarrow$ a function of scenario $\boldsymbol{t}$
- def. of worst scenario

$$\boldsymbol{t}^{\mathrm{w}} = \arg\min\{\lambda^*(\boldsymbol{t}) \mid t \in \mathcal{T}(\alpha;\tilde{\boldsymbol{t}})\}$$

- For a given $\boldsymbol{t}$,
  $\lambda^*(\boldsymbol{t})$ is defined by the classical upper bound principle.
- Find the severest scenario among. . .



ex.) $\alpha = 2$

classical limit analysis (truss)

- upper bound principle—linear programming (LP)

$$\lambda^*(\boldsymbol{t}) = \min_{\boldsymbol{c},\boldsymbol{u}} \quad -\boldsymbol{p}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{u} + \sum_{i=1}^{m} q_{\mathrm{y}i} c_i$$
$$\mathrm{s.\,t.} \quad (\boldsymbol{c}, \boldsymbol{u}) \in \text{(kinematically admissible)}$$

  - $\lambda^*(\boldsymbol{t})$: limit load factor
  - $\boldsymbol{u}$: nodal displacements
  - c_i : plastic member elongation
  - yield condition

$$|q_i| \le q_{\mathrm{y}i}$$

  - external load

$$\boldsymbol{p} = \boldsymbol{p}_{\mathrm{d}} + \lambda \boldsymbol{p}_{\mathrm{r}}$$

worst scenario detection using upper-bound principle

- upper bound principle

$$\lambda^*(\boldsymbol{t}) = \min_{\boldsymbol{c},\boldsymbol{u}} \quad -\boldsymbol{p}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{u} + \sum_{i=1}^{m} q_{\mathrm{y}i} c_i \quad \mathrm{s.\,t.} \quad (\boldsymbol{c},\boldsymbol{u}) \in \text{(kinematically admissible)}$$

- worst scenario problem

$$\lambda^*(\boldsymbol{t}^{\mathrm{w}}) = \min_{\boldsymbol{q}_{\mathrm{y}}} \quad \min_{\boldsymbol{c},\boldsymbol{u}} \quad -\boldsymbol{p}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{u} + \sum_{i=1}^{m} q_{\mathrm{y}i} c_i \quad \mathrm{s.\,t.} \quad (\boldsymbol{c},\boldsymbol{u}) \in \mathcal{A} \qquad (\clubsuit)$$

- yield force satisfies $q_{\mathrm{y}i} = \begin{cases} \tilde{q}_{\mathrm{y}i} & \text{member } i \text{ is present} \\ 0 & \text{member } i \text{ is absent} \end{cases}$

- $\rightarrow$ reformulate (♣) as a MIP problem

worst scenario problem

- original formulation:

$$\lambda^*(\boldsymbol{t}^{\rm w}) = \min_{\boldsymbol{q}_{\rm y}} \min_{(\boldsymbol{u},\boldsymbol{c})\in\mathcal{A}} -\boldsymbol{p}_{\rm d}^{\rm T}\boldsymbol{u} + \sum_{i=1}^{m} q_{{\rm y}i}c_i$$

($q_{{\rm y}i} = \tilde{q}_{{\rm y}i}$ or 0)

- MIP formulation:

$$\begin{aligned} \min_{(\boldsymbol{c},\boldsymbol{u})\in\mathcal{A},\boldsymbol{t}\in\mathcal{T}(\alpha,\tilde{\boldsymbol{t}}),\boldsymbol{w}} \quad & -\boldsymbol{p}_{\rm d}^{\rm T}\boldsymbol{u} + \sum_{i=1}^{m} w_i \\ \text{s. t.} \quad & M(1-t_i) \geq |w_i - \tilde{q}_{{\rm y}i}c_i|, \\ & Mt_i \geq |w_i| \end{aligned}$$

- $t_i = 0$ or 1 $\quad w_i = \begin{cases} \tilde{q}_{{\rm y}i}c_i & \text{if } t_i = 1 \\ 0 & \text{if } t_i = 0 \end{cases}$ $\quad$ ($M \gg 0$: constant)

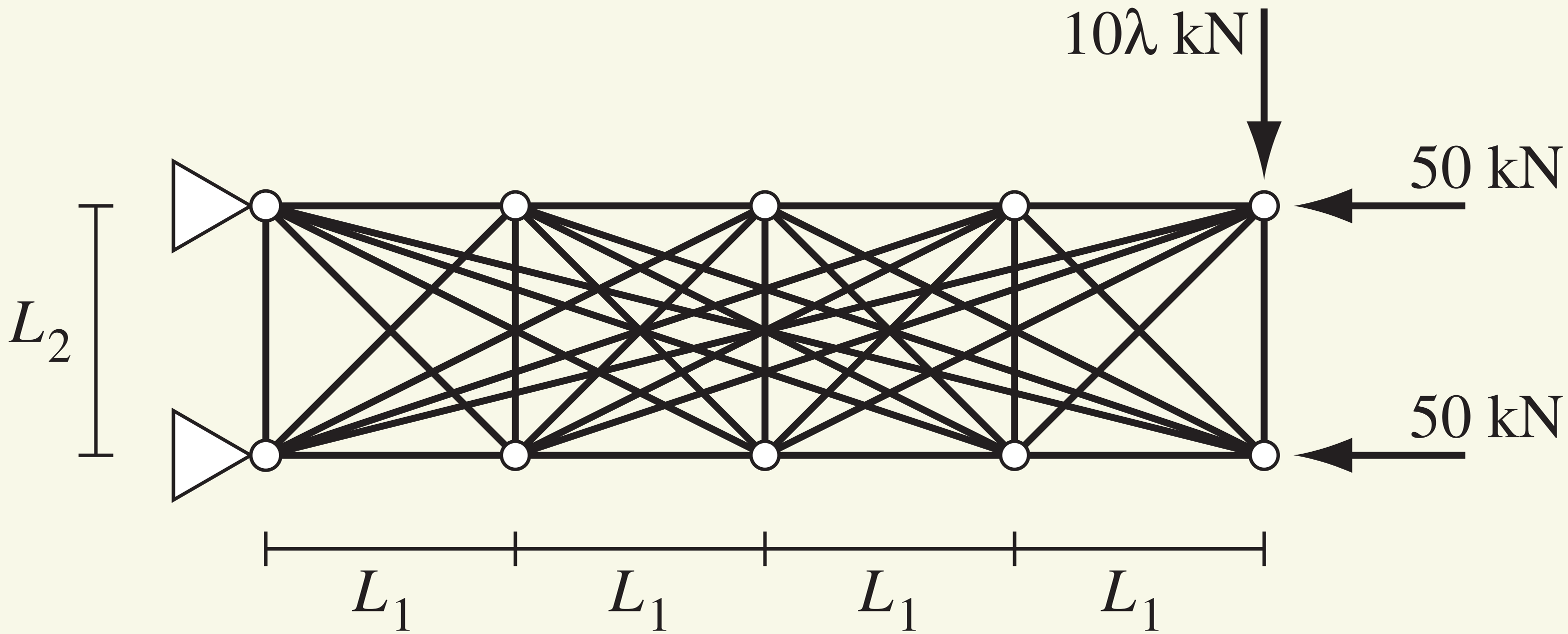
full formulation

- MIP formulation:

$$
\left.\begin{aligned}
\min_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{c},\boldsymbol{w}} \quad & -\boldsymbol{p}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{u} + \sum_{i=1}^{m} w_i \\
\mathrm{s.\,t.} \quad & \boldsymbol{p}_{\mathrm{r}}^{\mathrm{T}}\boldsymbol{u} = 1, \\
& -c_i \le \boldsymbol{h}_i^{\mathrm{T}}\boldsymbol{u} \le c_i, \\
& -M(1 - t_i) \le w_i - \tilde{q}_{\mathrm{y}i} c_i \le 0, \\
& 0 \le w_i \le M t_i, \\
& \sum_{i=1}^{m} (\tilde{t}_i - t_i) \le \alpha, \quad t_i \le 1, \\
& t_i \in \{0, 1\} \qquad (\diamondsuit)
\end{aligned}\right\} \qquad (\spadesuit)
$$

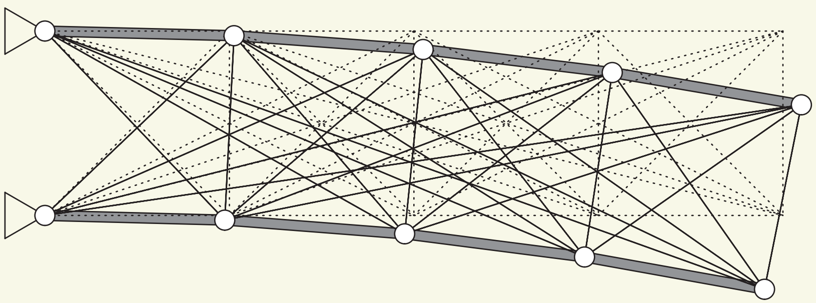
- $(\diamondsuit)$: integrality constraint
  The others are linear constraints.
- $(\spadesuit)$ can be solved globally
  by a branch-and-bound method, etc.

ex.) 32-bar truss



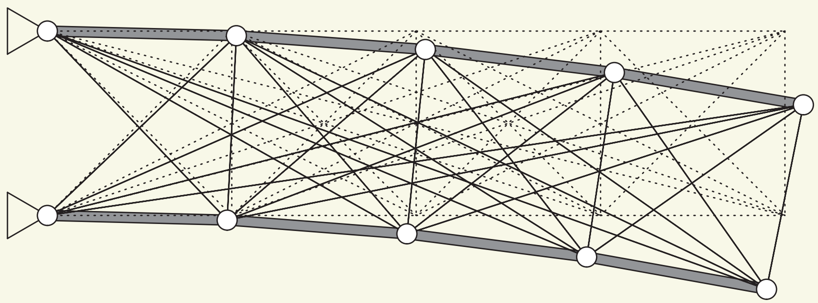
- $L_1 = L_2 = 1\,\mathrm{m}$
- $q_{\mathrm{y}i} = 200\,\mathrm{kN}$ (yield force)
- $\lambda(\tilde{\boldsymbol{x}}) = 10.0$ (nominal case: without damage)

ex.) 32-bar truss: worst scenarios

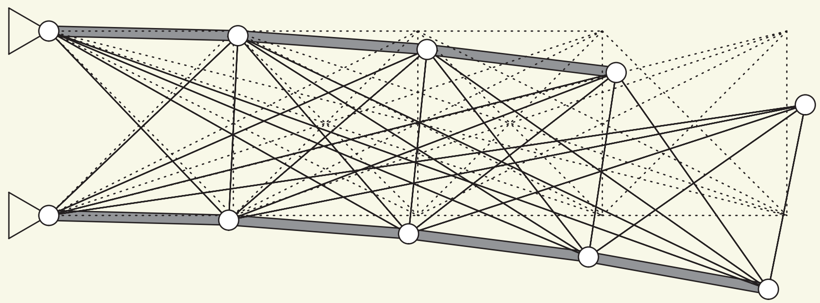


$$\lambda(\tilde{\boldsymbol{x}}) = 10.00$$

ex.) 32-bar truss: worst scenarios



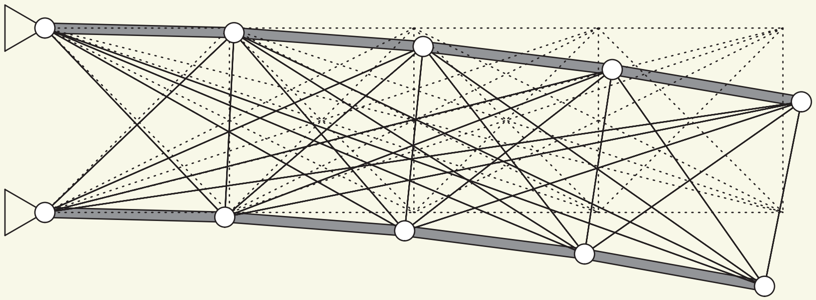
$\lambda(\tilde{\boldsymbol{x}}) = 10.00$



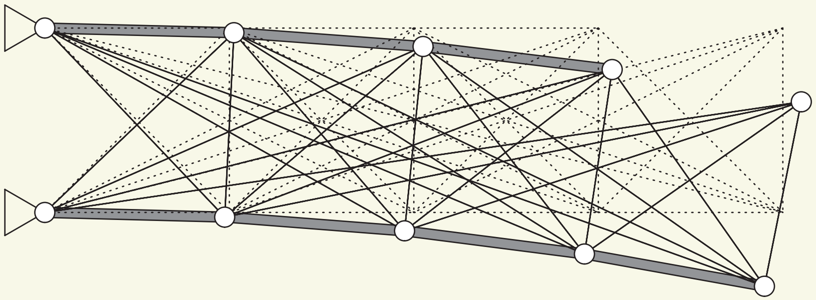
$\alpha = 1,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 8.750$

- thick line: yielding member
- damaged member: absent

ex.) 32-bar truss: worst scenarios



$\lambda(\tilde{\boldsymbol{x}}) = 10.00$



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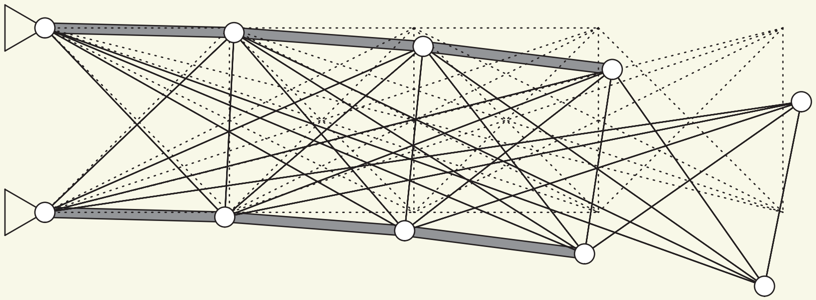
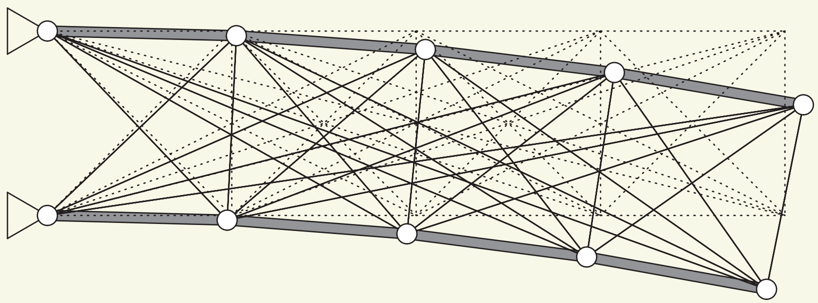
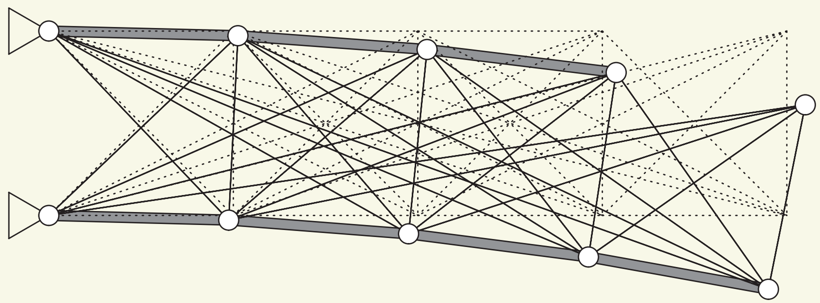


$\alpha = 2,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 7.500$

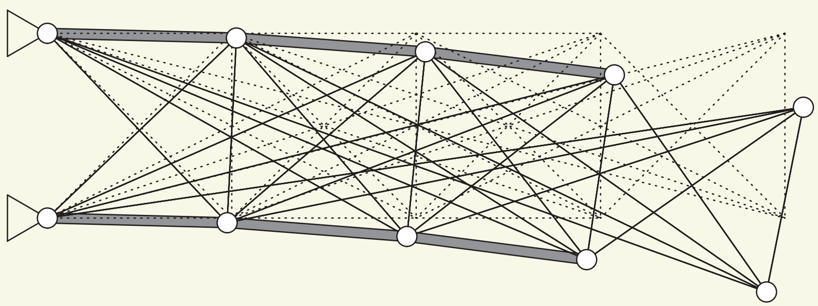
ex.) 32-bar truss: worst scenarios



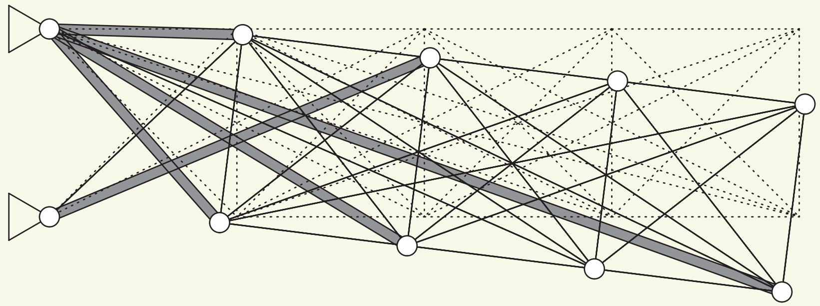
$\lambda(\tilde{\boldsymbol{x}}) = 10.00$



$\alpha = 1,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 8.750$



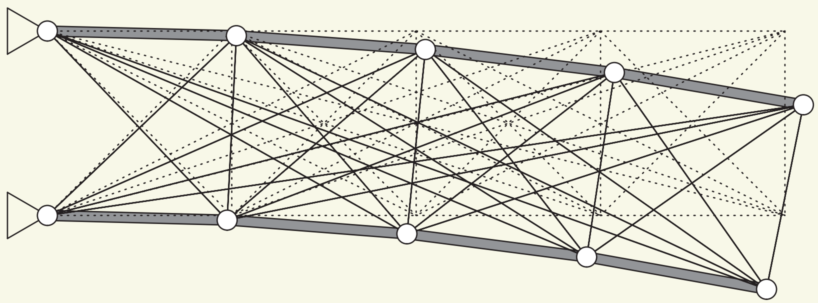
$\alpha = 2,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 7.500$



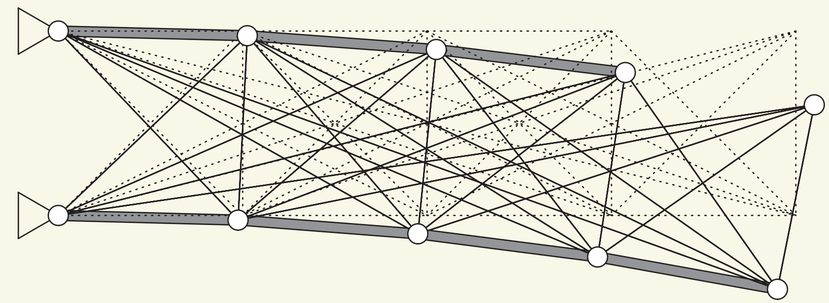
$\alpha = 3,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 6.072$

(damaged members at $\alpha = 3$)
$\not\supseteq$ (damaged members at $\alpha = 2$)

ex.) 32-bar truss: worst scenarios



$\lambda(\tilde{\boldsymbol{x}}) = 10.00$



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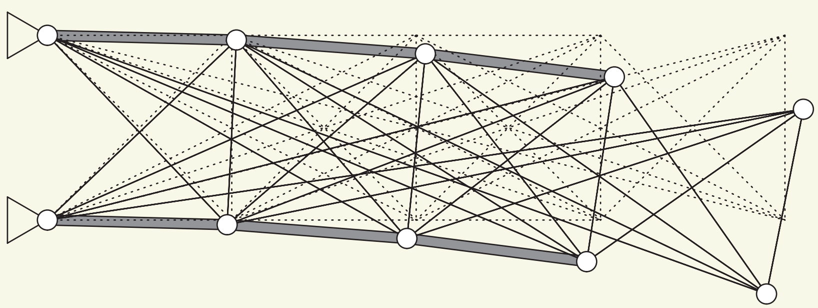

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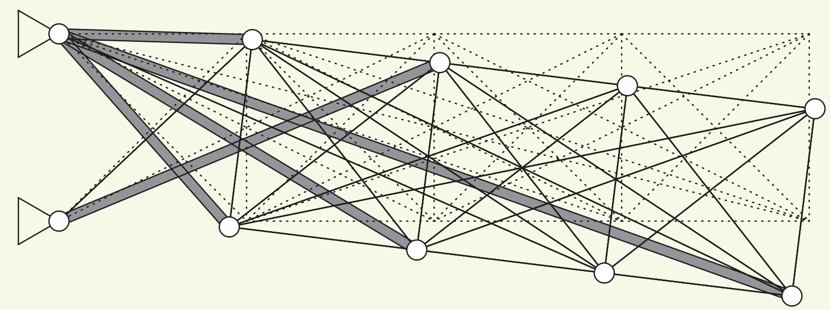
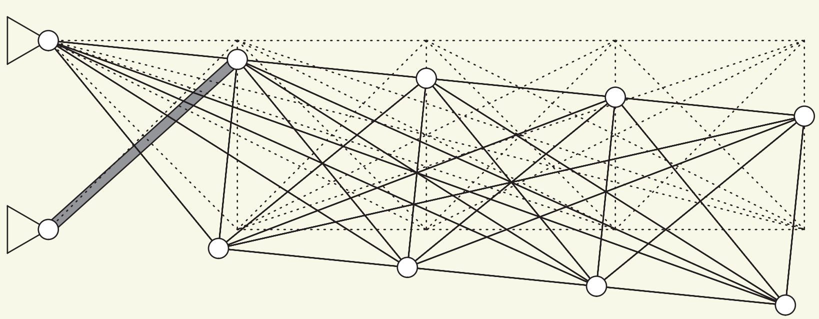


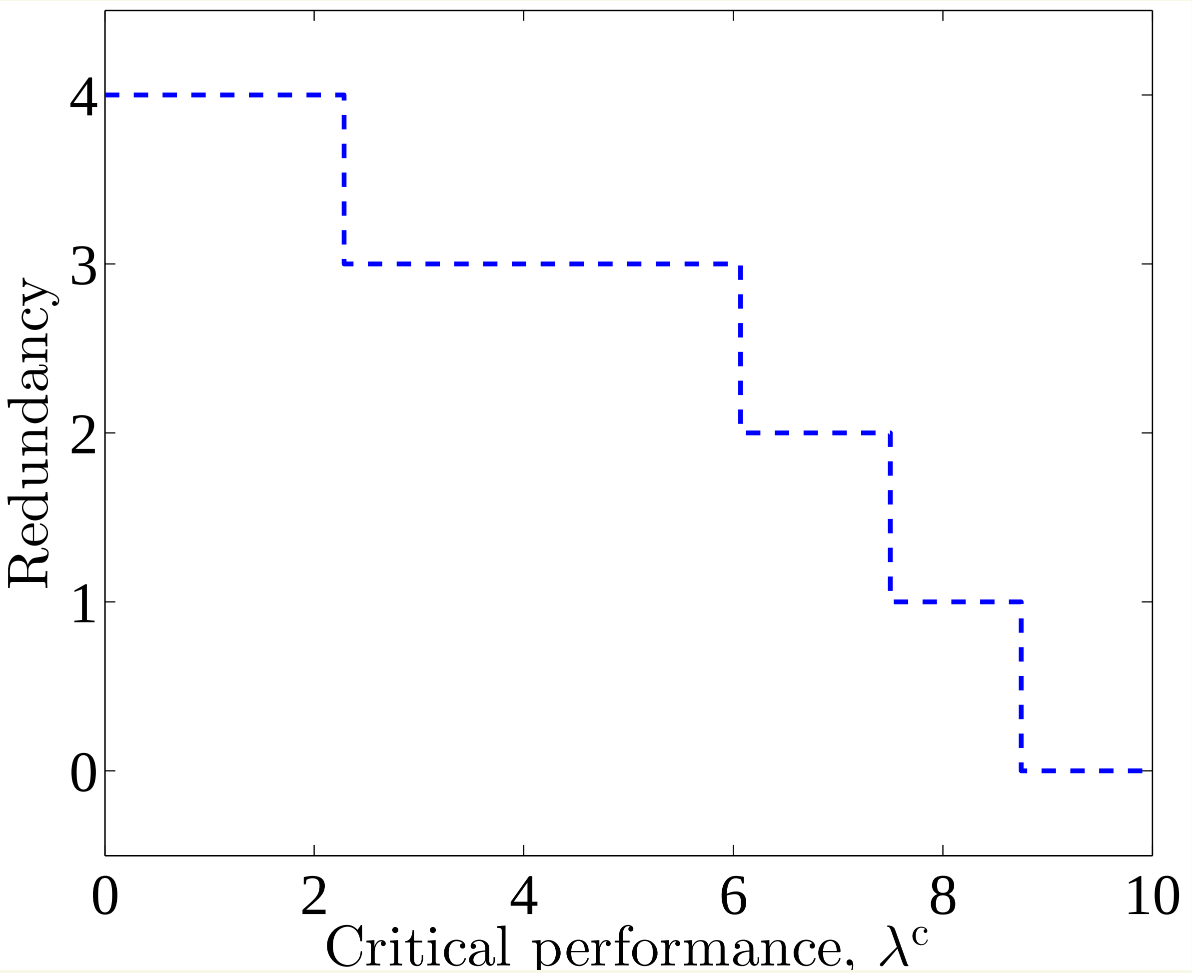
$\alpha = 3,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 6.072$



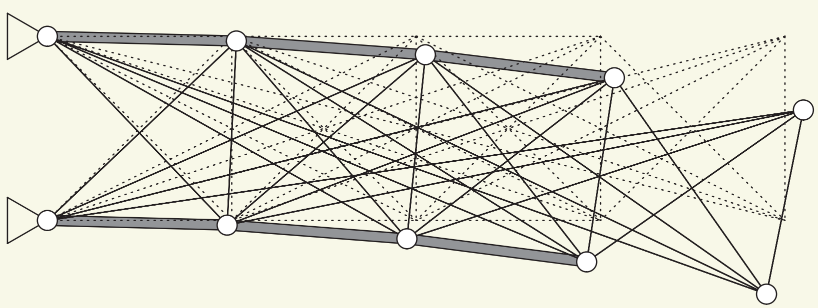
$\leftarrow \alpha = 4,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 2.286$

ex.) 32-bar truss: redundancy curve

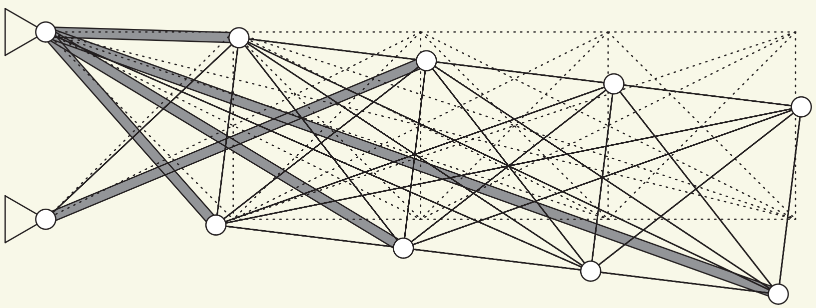
- redundancy vs. performance (bound for limit load factor)



ex.) 32-bar truss: deficiency set depends on α



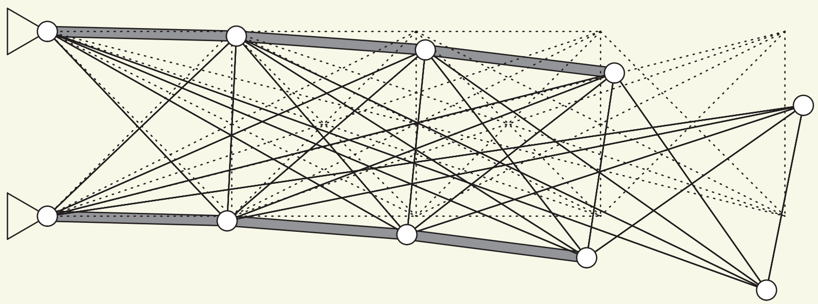
$\alpha = 2,\ \lambda(\boldsymbol{x}^{\rm worst}) = 7.500$



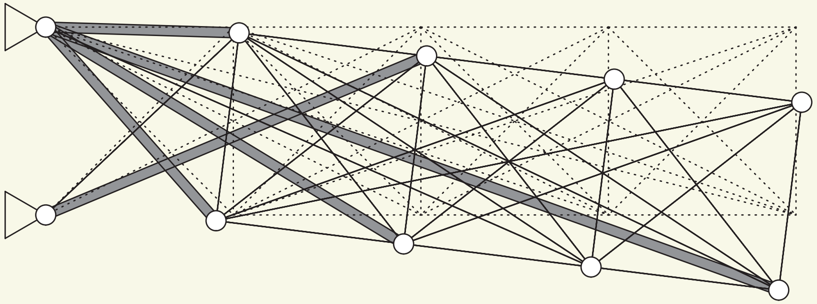
$\alpha = 3,\ \lambda(\boldsymbol{x}^{\rm worst}) = 6.072$

- (damaged members at $\alpha = 2$) $\not\subseteq$ (damaged members at $\alpha = 3$)
-

ex.) 32-bar truss: deficiency set depends on α

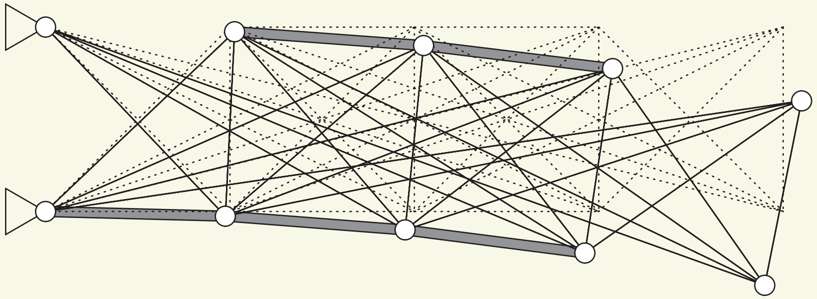


$\alpha = 2,\ \lambda(\boldsymbol{x}^{\text{worst}}) = 7.500$



$\alpha = 3,\ \lambda(\boldsymbol{x}^{\text{worst}}) = 6.072$

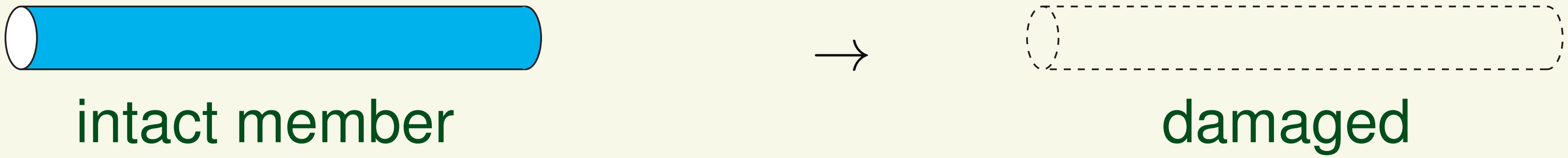
- (damaged members at $\alpha = 2$) $\not\subseteq$ (damaged members at $\alpha = 3$)
- If "$\subseteq$", the worst scenario is



$6.250 >$ (truly worst value)

ex.) 32-bar truss: partial deficiency model

- complete deficiency model (discussed)

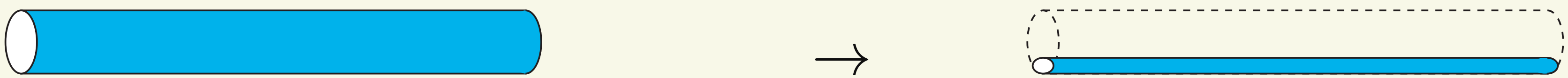


ex.) 32-bar truss: partial deficiency model

- complete deficiency model (discussed)

intact member damaged

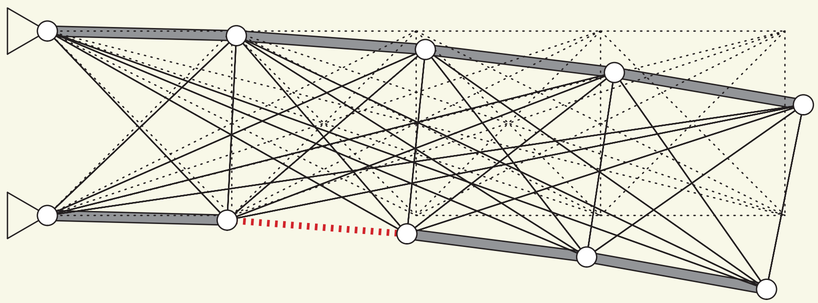
- partial deficiency model (new)



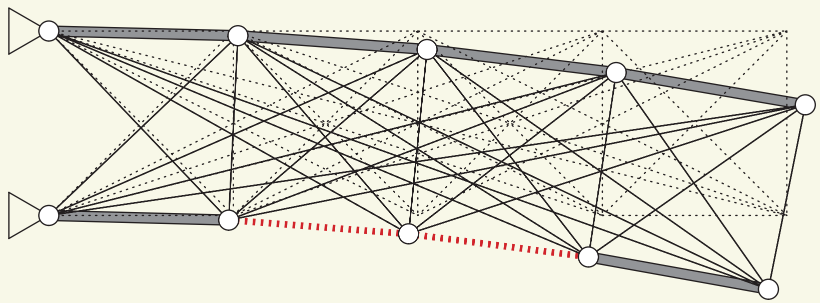
intact member damaged

  - cross-sectional area $x_i = \tilde{x}_i \ \rightarrow \ x_i = \rho \tilde{x}_i$
    $\rho \in (0, 1)$ (constant)
  - MIP formulation is also available

ex.) 32-bar truss: partial deficiency model



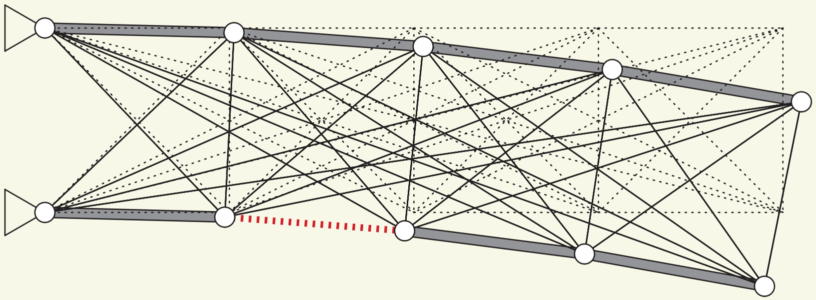
$\alpha = 1,\ \lambda^* = 9.000$



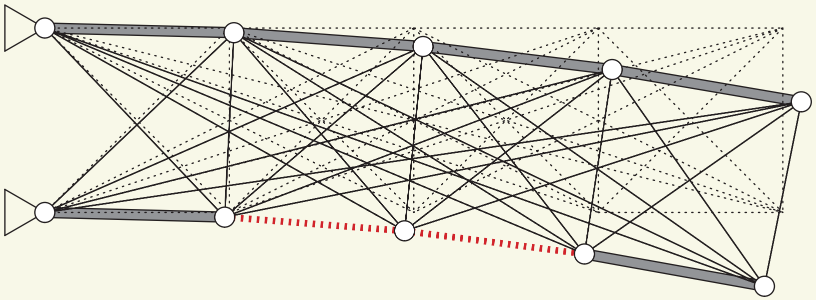
$\alpha = 2,\ \lambda^* = 8.000$

- thick line: yielding member
- damaged member: cross section is reduced by 80%

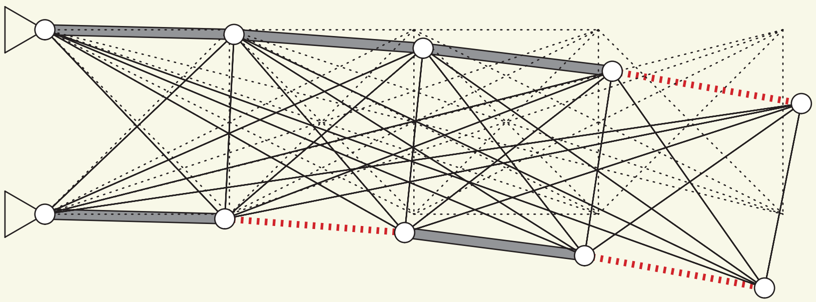
ex.) 32-bar truss: partial deficiency model



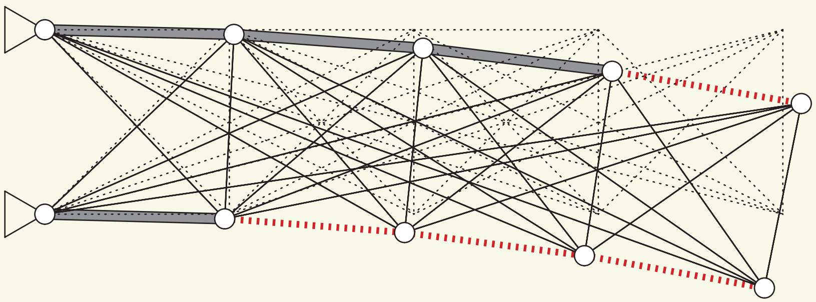
$\alpha = 1,\ \lambda^* = 9.000$



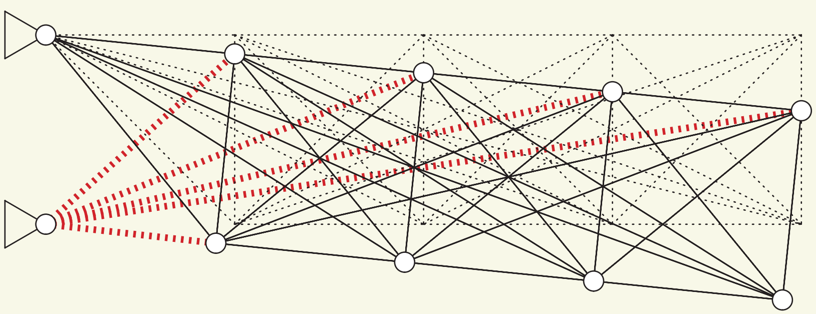
$\alpha = 2,\ \lambda^* = 8.000$



$\alpha = 3,\ \lambda^* = 7.000$

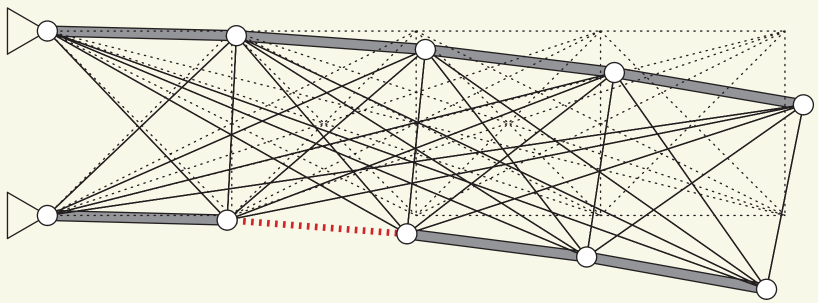


$\alpha = 4,\ \lambda^* = 6.000$

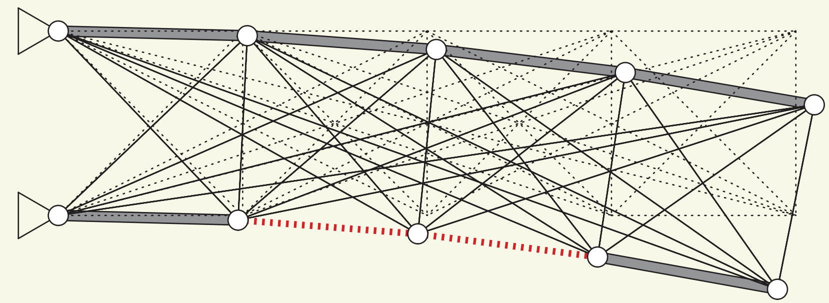


$\leftarrow \alpha = 5,\ \lambda^* = 3.270$

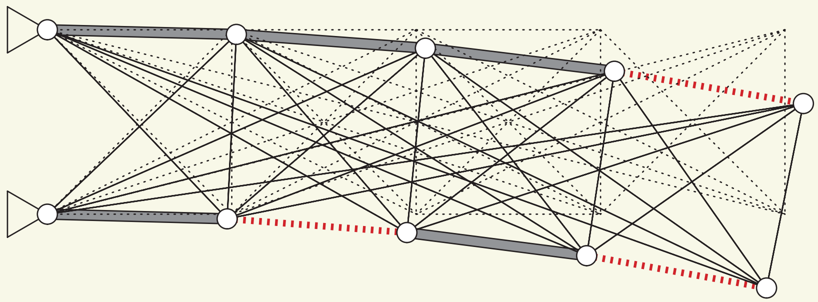
ex.) 32-bar truss: partial deficiency model



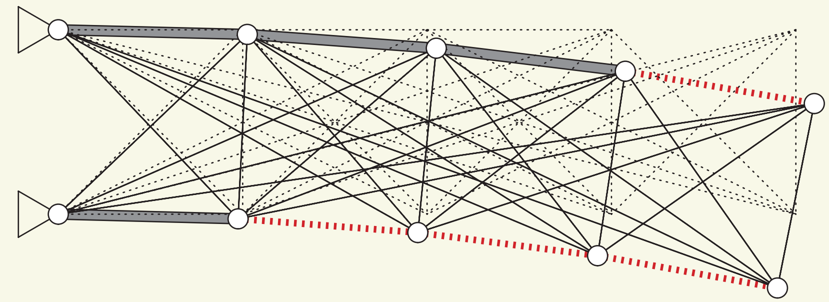
$\alpha = 1,\ \lambda^* = 9.000$



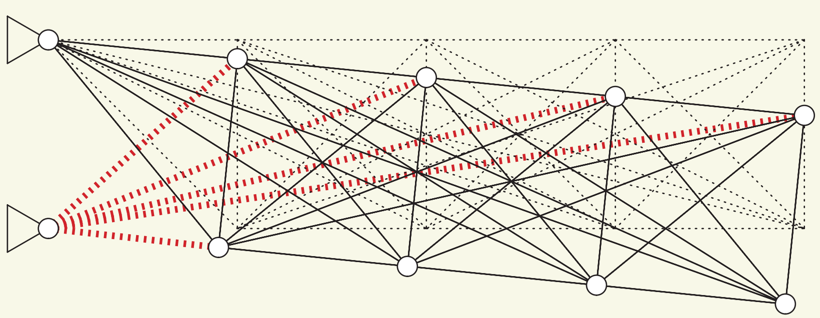
$\alpha = 2,\ \lambda^* = 8.000$



$\alpha = 3,\ \lambda^* = 7.000$

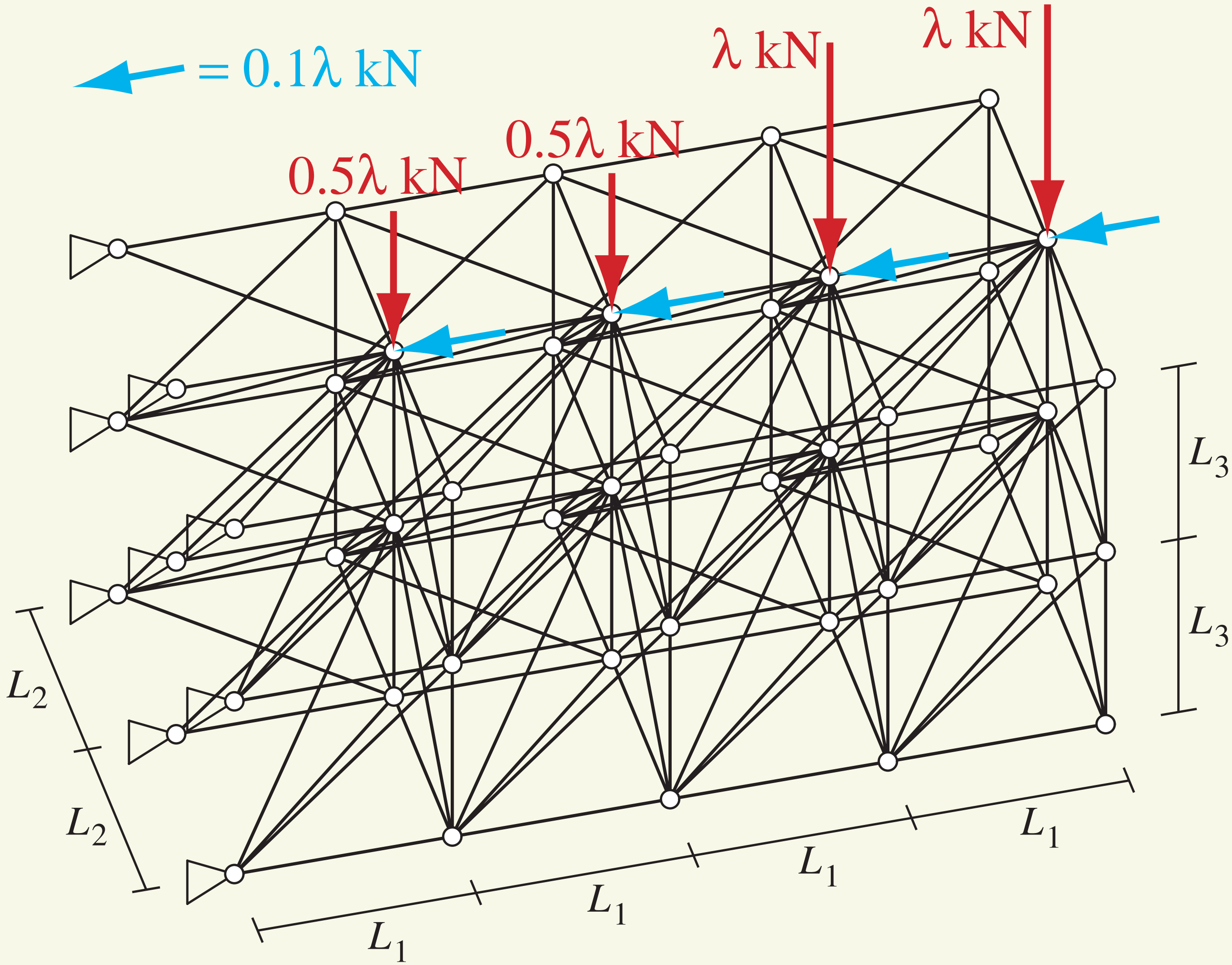


$\alpha = 4,\ \lambda^* = 6.000$



Worst scenario depends on α

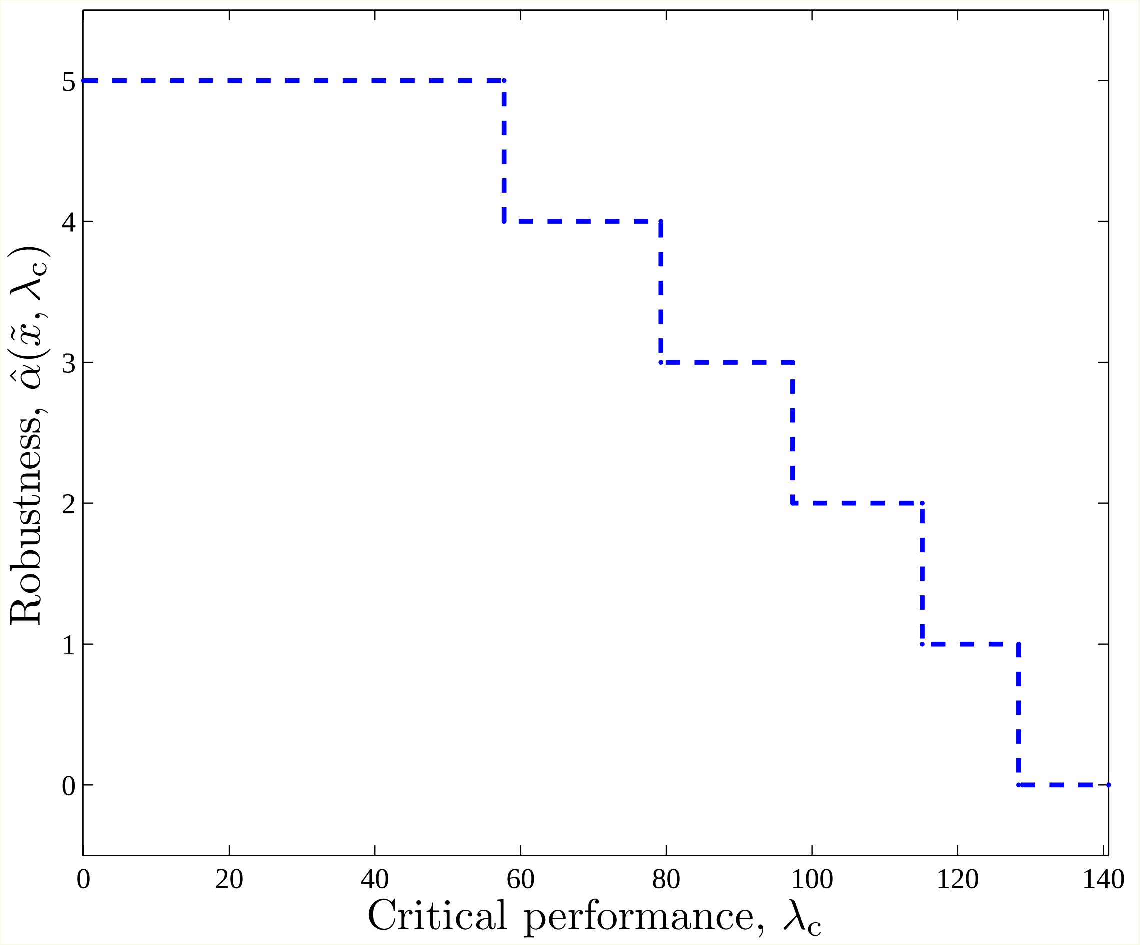
ex.) 164-bar space truss



- $L_1 = L_2 = L_3 = 1\,\mathrm{m}$
- $q_{\mathrm{y}i} = 100\,\mathrm{kN}$ (yield force)

ex.) space truss: redundancy curve

- redundancy vs. performance (bound for limit load factor)



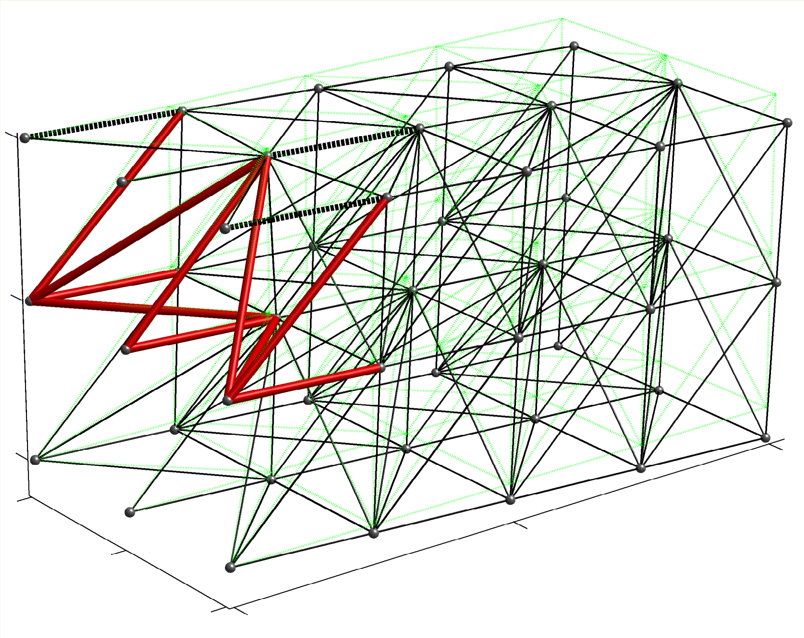
ex.) space truss: computational time

α	$\lambda(\boldsymbol{x}^{\rm worst})$	CPU (s)
0	140.7079	0.1
1	128.3622	0.7
2	115.1253	33.5
3	97.3447	570.4
4	79.2562	2,455.6
5	57.7350	2,872.7
6	0.0000	25.5

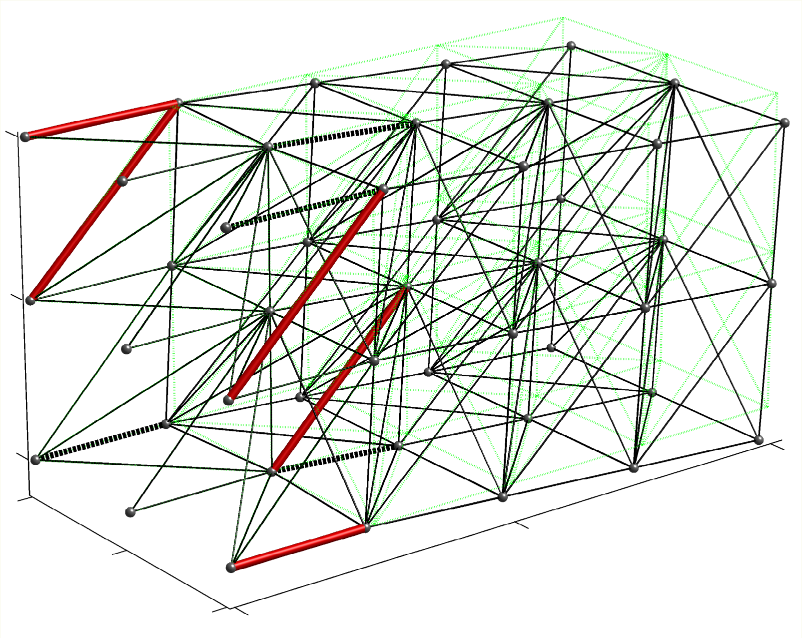
- CPLEX Ver. 11.2 on Core 2 Duo (2.26 GHz)
- # of scenarios at $\alpha = 5$

$$1 + \sum_{\alpha=1}^{5} \binom{m}{\alpha} = 959{,}418{,}328$$

ex.) 32-bar truss: worst scenarios



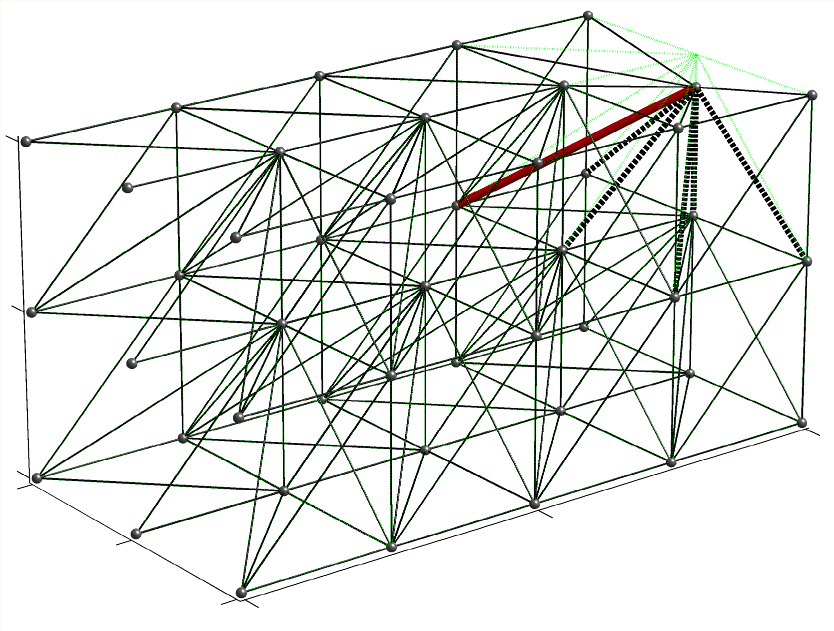
$\alpha = 3,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 97.375$



$\alpha = 4,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 79.256$

- yield members depend on α

ex.) 32-bar truss: worst scenarios



$$\alpha = 5,\ \lambda(\boldsymbol{x}^{\mathrm{worst}}) = 57.535$$

- local collapse mode
  - collapse mode depends on α

conclusions

- structural redundancy
  - robustness against uncertainty in damage
- worst scenario in limit analysis
  - uncertainty in deficiency of structural component
    - given: # of deficient components, α
    - w. s. minimizes the limit load factor
  - key components (which cause the w. s.) depend on α
- computational method
  - mixed integer programming formulation
  - global optimization