Second-Order Cone Programming for Complementary Energy Principle of Cable Networks

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Total potential energy principle :

- contains only displacement
- stable equilibrium $\Longrightarrow$ min. TPE

Complementary energy principle :

- continuum
  - 2nd PK stress, displacement (Hellinger, '14)
  - 1st PK stress, Biot stress, rotation (Koiter, '76)
  - 1st PK stress* (Zubov, '70; Koiter, '76)
- truss
  - internal force vector* (Mikkola, '89)
- stable equilibrium $\not\Rightarrow$ min. CE

* multi-valued function

- "1st PK stress – displacement gradient" (constitutive law)
  - multi-valued inversion (Koiter,'76)

Questions :

1. Does CE principle contain only stress components?

2. Does CE always attain minimum at the stable equilibrium?

3. Is CE determined uniquely?
   (Otherwise, CE may be a multi-valued function.)

4. Is CE formulated in an explicit algebraic form?

Questions :

1. Does CE principle contain only stress components?

2. Does CE always attain minimum at the stable equilibrium?

3. Is CE determined uniquely?

   (Otherwise, CE may be a multi-valued function.)

4. Is CE formulated in an explicit algebraic form?

Our results :

— 'yes' for cable networks

which can transmit only tension force

CE principle for trusses :

small deformation :

$$(\Pi^{\mathrm{C}}) : \min \sum_{i=1}^{n^m} w_i^{\mathrm{C}}(q_i) - \boldsymbol{x}_0^\top \boldsymbol{q}$$

$$\text{s.t.} \quad \boldsymbol{A}\boldsymbol{q} + \boldsymbol{f} = \boldsymbol{0}.$$: equilibrium eqs.

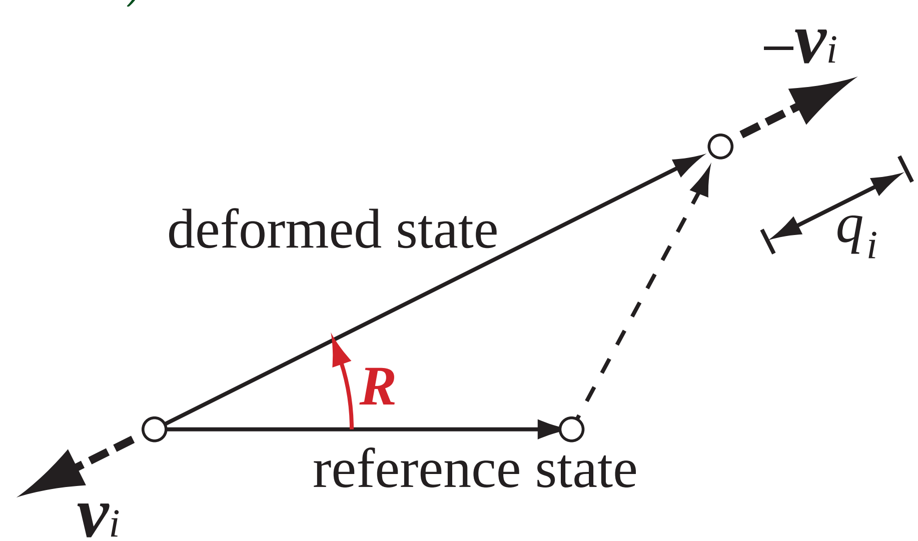
finite deformation :

Equilibrium eqs. : $\boldsymbol{R}\boldsymbol{A}\boldsymbol{q} + \boldsymbol{f} = \boldsymbol{0}.$ — unknown rotation

Mikkola's CE ('89) :

$$\Pi^{\mathrm{C}}(\boldsymbol{v}) = \sum_{i=1}^{n^m} \left(w_i^{\mathrm{C}}(\|\boldsymbol{v}_i\|) \pm l_i^0 \|\boldsymbol{v}_i\| - \boldsymbol{b}_i^\top \boldsymbol{v}_i \right)$$

— only stationary principle



- $q_i \in \mathbf{R}$: axial force
- $\boldsymbol{v}_i \in \mathbf{R}^3$: internal force vector

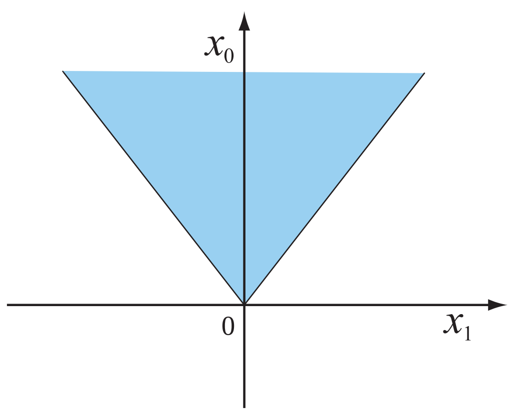
Second-Order Cone Programming : SOCP

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad x_0 \ge \|\boldsymbol{x}_1\|. \end{aligned}$$

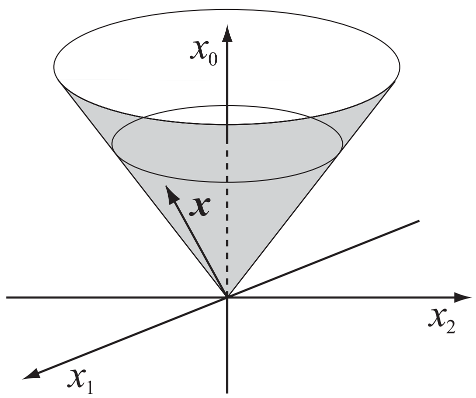
$\|\boldsymbol{x}_1\| = (\boldsymbol{x}_1^\top \boldsymbol{x}_1)^{1/2}$: Euclidean norm

$(\text{SOC}) = \{(x_0, \boldsymbol{x}_1) | x_0 \ge \|\boldsymbol{x}_1\|\}$: second-order cone

1. convex nonlinear programming
2. including LP, QP, etc.
3. primal-dual interior-point method (Monteiro & Tsuchiya, '00)
4. applications
   - truss topology optimization (Jarre *et al.*, '98)



SOC in 2D ($x_0 \ge |x_1|$)



SOC in 3D ($x_0 \ge \sqrt{x_{11}^2 + x_{12}^2}$)

Second-Order Cone Programming : SOCP

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad x_0 \geq \|\boldsymbol{x}_1\|. \end{array}$$

min. of TPE $\Longrightarrow$ primal SOCP

$\Updownarrow$ Lagrangian duality

min. of CE $\Longleftarrow$ dual SOCP

Min. of TPE :

$$(\Pi): \min \quad \Pi = \sum_{i=1}^{n^m} w_i(y_i) - \boldsymbol{f}^\top \boldsymbol{u}$$
$$\text{s.t.} \quad y_i = \|\mathcal{B}_i \boldsymbol{u}\|.$$

Variables: y_i : member elongation $\quad$ $\boldsymbol{u}$: disp. of internal nodes

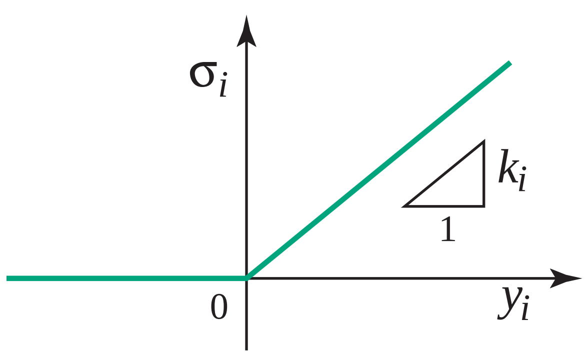
Given: $\boldsymbol{f}$: external forces $\quad$ l_i^0 : member unstressed length

strain energy :

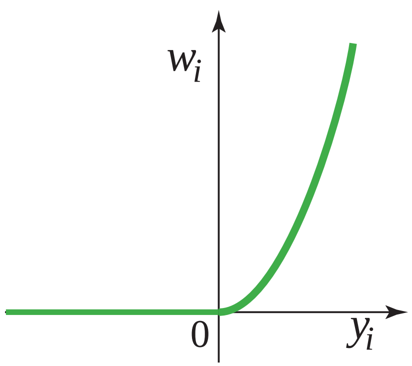
$$w_i(y_i) = \begin{cases} \frac{1}{2}k_i y_i^2 & (y_i \geq 0), \\ 0 & (y_i < 0). \end{cases}$$

compatibility :

$$y_i = \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i^0\| - l_i^0$$



axial force–elongation



strain energy

min. of TPE $\Longrightarrow$ primal SOCP

$\Updownarrow$ Lagrangian duality

min. of CE $\Longleftarrow$ dual SOCP

SOCP formulation :

1. convex relaxation

   - compatibility :

$$y_i = \|\mathcal{B}_i \boldsymbol{u}\| \qquad \Longrightarrow \qquad y_i \geq \|\mathcal{B}_i \boldsymbol{u}\|$$: second-order cone

2. strain energy

$$w_i(y_i)$$: for cable $\Longrightarrow$ $$\frac{1}{2} k_i y_i^2$$: for truss

SOCP formulation :

1. convex relaxation

   - compatibility :

$$y_i = \|\mathcal{B}_i \boldsymbol{u}\| \qquad \Longrightarrow \qquad y_i \geq \|\mathcal{B}_i \boldsymbol{u}\|$$

: second-order cone

2. strain energy

$$w_i(y_i) \quad \text{: for cable} \qquad \Longrightarrow \qquad \frac{1}{2} k_i y_i^2 \quad \text{: for truss}$$

$$\begin{aligned} \text{(P)}: \ \min \quad & \phi_P = \sum_{i=1}^{n^m} \frac{1}{2} k_i y_i^2 - \boldsymbol{f}^\top \boldsymbol{u} \\ \text{s.t.} \quad & y_i \geq \|\mathcal{B}_i \boldsymbol{u}\|. \end{aligned}$$

1. (P) = SOCP

2. (P) has the same optimizer as that of min. of TPE

3. polynomial-time algorithm — IPM

- Y. Kanno, M. Ohsaki and J. Ito ('02)
  Int. J. Numer. Meth. Engng., Vol. 55(9) 1079–1114.

$$\begin{array}{ccc} \text{min. of TPE} & \Longrightarrow & \text{primal SOCP} \\ & & \Updownarrow \quad \text{Lagrangian duality} \\ \text{min. of CE} & \Longleftarrow & \text{dual SOCP} \end{array}$$

LP (Linear Programming) :

$$\begin{aligned} \text{(P)}: \ \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Lagrangian :

$$L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) = \begin{cases} \boldsymbol{c}^\top \boldsymbol{x} - \boldsymbol{y}^\top (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) - \boldsymbol{s}^\top \boldsymbol{x} & (\boldsymbol{s} \geq \boldsymbol{0}), \\ +\infty & (\text{otherwise}). \end{cases}$$

LP (Linear Programming) :

$$
\begin{aligned}
(\mathrm{P}):\ \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}.
\end{aligned}
$$

Lagrangian :

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$$

Lagrangian dual :

$$
\begin{aligned}
(\mathrm{D}):\ \max \quad & \boldsymbol{b}^\top \boldsymbol{y} \\
\text{s.t.} \quad & \boldsymbol{A}^\top \boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}, \quad \boldsymbol{s} \geq \boldsymbol{0}.
\end{aligned}
$$

"Nice" duality :

1. (D) is also LP
2. (D) does not contain "$\boldsymbol{x}$"
3. strong duality : $\min(\mathrm{P}) = \max(\mathrm{D})$

not satisfied by NLP in general

SOCP :

$$\text{(P)}: \min \ \boldsymbol{c}^\top \boldsymbol{x} \quad \text{s.t.} \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \in \text{SOC}.$$

(Extended) Lagrangian :

$$L(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) = \begin{cases} \boldsymbol{c}^\top \boldsymbol{x} - \boldsymbol{y}^\top (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) - \boldsymbol{s}^\top \boldsymbol{x} & (\boldsymbol{s} \in \text{SOC}), \\ +\infty & \text{(otherwise)}. \end{cases}$$

Lagrangian dual :

$$\text{(D)}: \max \ \boldsymbol{b}^\top \boldsymbol{y} \quad \text{s.t.} \ \boldsymbol{A}^\top \boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}, \quad \boldsymbol{s} \in \text{SOC}.$$

"Nice" duality :

1. (D) is also SOCP
2. (D) does not contain "$\boldsymbol{x}$"
3. strong duality : $\min \text{(P)} = \max \text{(D)}$

1. self-duality :
   $\boldsymbol{s} \in \text{SOC} \iff \boldsymbol{s}^\top \boldsymbol{x} \geq 0 \ (\forall \boldsymbol{x} \in \text{SOC})$
2. L is linear w.r.t. $\boldsymbol{x}$
   (though (P) is non-linear)

SOCP formulation :

$$
\begin{array}{lll}
(\mathrm{P}) : & \min & \phi_P(\boldsymbol{y}, \boldsymbol{u}) \\
 & \text{s.t.} & y_i \geq \|\mathcal{B}_i \boldsymbol{u}\|.
\end{array}
$$

(Extended) Lagrangian :

$$
L(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{q}, \boldsymbol{v}) = \begin{cases} \phi_P(\boldsymbol{y}, \boldsymbol{u}) - \displaystyle\sum_{i=1}^{n^m} (q_i y_i + \boldsymbol{v}_i^\top \mathcal{B}_i \boldsymbol{u}) & (q_i \geq \|\boldsymbol{v}_i\| \ (\forall i)), \\ +\infty & (\text{otherwise}). \end{cases}
$$

- linear w.r.t. $\boldsymbol{u}$

SOCP formulation :

$$(\mathrm{P}): \min \quad \phi_P(\boldsymbol{y}, \boldsymbol{u}) \quad \text{s.t.} \quad y_i \geq \|\mathcal{B}_i \boldsymbol{u}\|.$$

(Extended) Lagrangian :

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- linear w.r.t. $\boldsymbol{u}$

Min. of TPE :

$$(\Pi): \min \quad \Pi(\boldsymbol{y}, \boldsymbol{u}) \quad \text{s.t.} \quad y_i = \|\mathcal{B}_i \boldsymbol{u}\|.$$

Classical Lagrangian (Hellinger '14; Washizu '82):

$$L_0(\boldsymbol{c}, \boldsymbol{u}, \boldsymbol{q}) = \Pi(\boldsymbol{y}, \boldsymbol{u}) - \sum_{i=1}^{n^m} q_i (y_i - \|\mathcal{B}_i \boldsymbol{u}\|).$$

- dual problem contains $\boldsymbol{u}$

min. of TPE $\Longrightarrow$ primal SOCP

$\Updownarrow$ Lagrangian duality

min. of CE $\Longleftarrow$ dual SOCP

$$(\Pi):\ \min\ \Pi(\boldsymbol{y},\boldsymbol{u}) \quad \text{s.t.}\ y_i = \|\mathcal{B}_i\boldsymbol{u}\|.$$

$$(\Pi^{\mathrm{C}}):\ \min\ \Pi^{\mathrm{C}}(\boldsymbol{v}) \quad : \text{CE}$$

$$\text{s.t.}\ \sum_{i=1}^{n^m} \boldsymbol{B}_i^\top \boldsymbol{v}_i + \boldsymbol{f} = \boldsymbol{0}. \quad : \text{equilibrium eqs.}$$

$$\Pi^{\mathrm{C}}(\boldsymbol{v}) = \sum_{i=1}^{n^m} \frac{\boldsymbol{v}_i^\top \boldsymbol{v}_i}{2k_i} + \sum_{i=1}^{n^m} l_i^0 \|\boldsymbol{v}_i\| + \sum_{i=1}^{n^m} (\boldsymbol{h}_i^0 - \boldsymbol{b}_i)^\top \boldsymbol{v}_i$$

Strong duality :

$\exists \boldsymbol{v}$ satisfying "equilibrium eqs." $\Longrightarrow$

(i) $\exists$ optimal solutions of (Π) and (Π^{C})

(ii) TPE$^* = -$CE*

(iii) $\boldsymbol{v}_i^*$ is optimum $\Longleftrightarrow$ KKT conditions

- (iii) $\Longleftrightarrow$ $\boldsymbol{v}_i^*$: internal force vector at the equilibrium uniquely exists

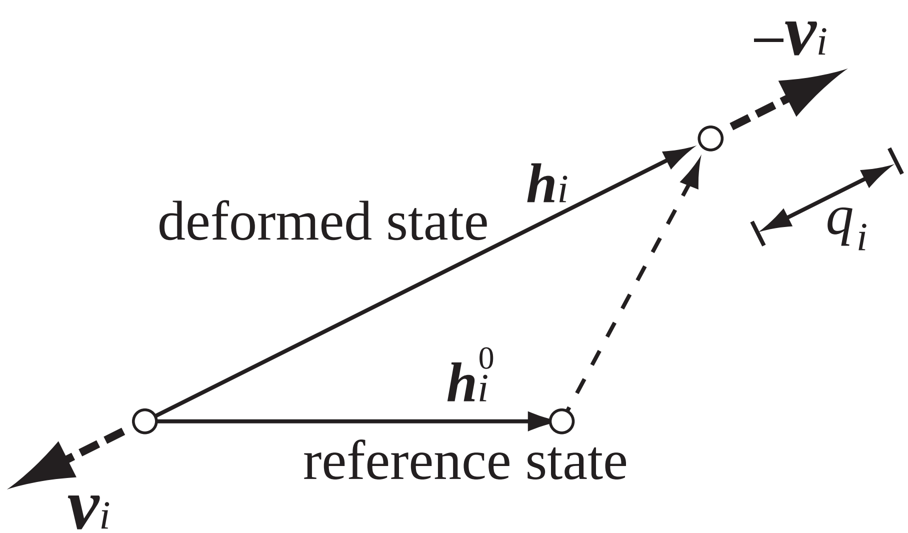
$$(\Pi): \min \quad \Pi(\boldsymbol{y}, \boldsymbol{u}) \quad \text{s.t.} \quad y_i = \|\mathcal{B}_i \boldsymbol{u}\|.$$

$$(\Pi^{\mathrm{C}}): \min \quad \Pi^{\mathrm{C}}(\boldsymbol{v}) \qquad : \text{CE}$$
$$\text{s.t.} \quad \sum_{i=1}^{n^m} \boldsymbol{B}_i^\top \boldsymbol{v}_i + \boldsymbol{f} = \boldsymbol{0}. \qquad : \text{equilibrium eqs.}$$

$$\Pi^{\mathrm{C}}(\boldsymbol{v}) = \sum_{i=1}^{n^m} \frac{\boldsymbol{v}_i^\top \boldsymbol{v}_i}{2k_i} + \sum_{i=1}^{n^m} l_i^0 \|\boldsymbol{v}_i\| + \sum_{i=1}^{n^m} (\boldsymbol{h}_i^0 - \boldsymbol{b}_i)^\top \boldsymbol{v}_i$$

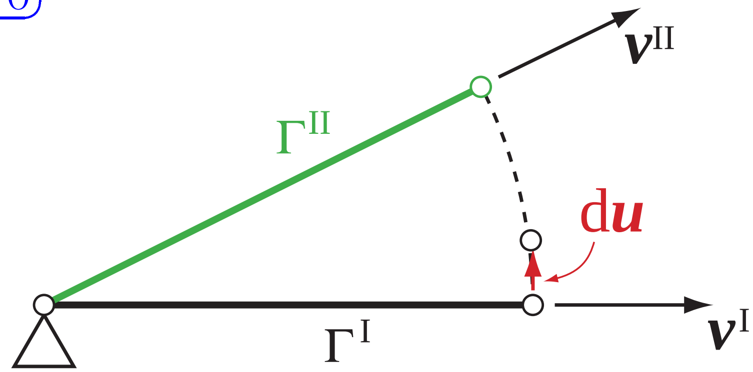
1. (Π^{C}) contains only $\boldsymbol{v}_i \in \mathbf{R}^3$ (= internal force vector)
2. CE attains minimum at the equilibrium
3. CE is determined uniquely

- CE depends on the reference state



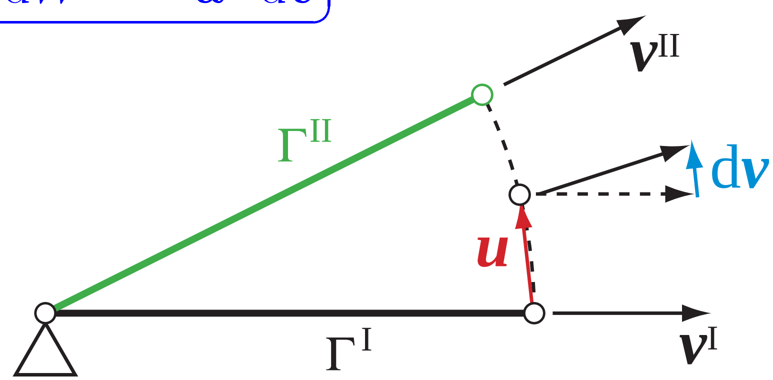
Rigid body rotation

work : $\mathrm{d}W = \boldsymbol{v}^\top \mathrm{d}\boldsymbol{u} = 0$



$$\boldsymbol{v}^\top \mathrm{d}\boldsymbol{u} = 0$$

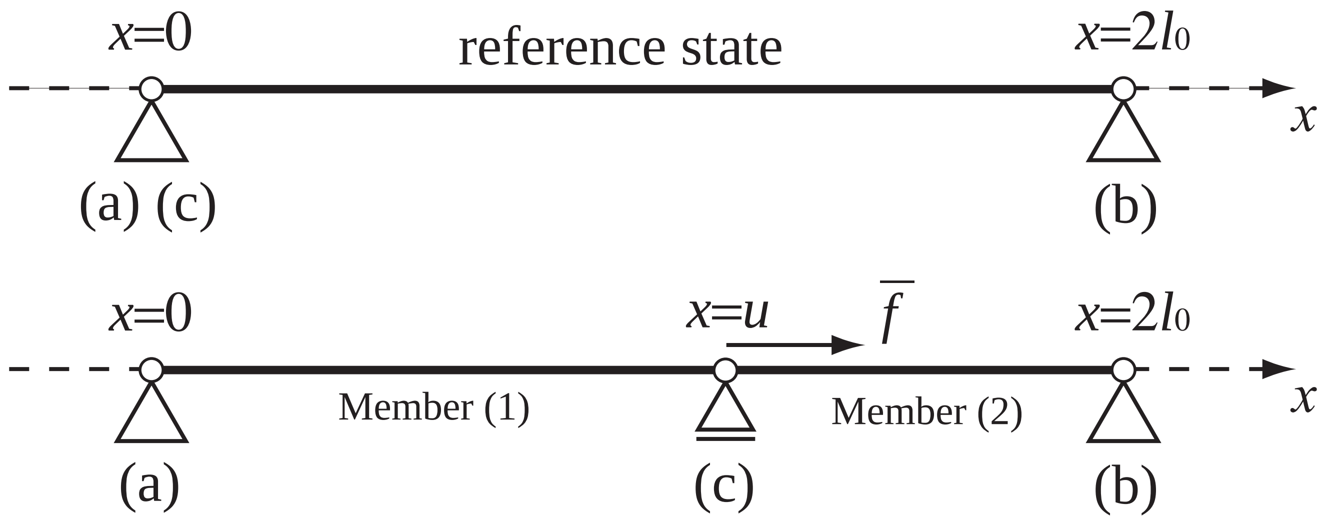
complementary work : $\mathrm{d}W^C = \boldsymbol{u}^\top \mathrm{d}\boldsymbol{v}$



$$\boldsymbol{u}^\top \mathrm{d}\boldsymbol{v} \neq 0$$

- $\Gamma^{\mathrm{I}} \Longrightarrow \Gamma^{\mathrm{II}}$: $\quad W^C \neq 0$
- CE depends on the rigid body rotation

Example :



$$(\Pi^{\mathrm{C}}) : \min \quad \Pi^{\mathrm{C}} = \frac{1}{2k}(v_1^2 + v_2^2) + l^0(|v_1| + |v_2|) - 2l^0 v_2$$
$$\text{s.t.} \quad v_1 + v_2 + f = 0.$$

- optimal solution : $(v_1^*, v_2^*) = (-f, 0)$ — member (2) is slackening

CE for small deformation : $\Pi^{\mathrm{C}}_{\mathrm{LIN}} = \frac{1}{2k}(v_1^2 + v_2^2) - 2l^0 v_2$

- optimal solution : $(v_1^*, v_2^*) = \left(-kl^0 + \frac{3f}{2},\ kl^0 - \frac{f}{2}\right)$

Conclusions :

1. CE principle for cable networks
   - min. of TPE — primal SOCP
   - min. of CE — dual SOCP
   - contains only stress components
2. CE function for large deformation
   - uniquely determined
   - formulated explicitly
   - always attains minimum at the equilibrium
   - agrees with the complementary work
3. Strong duality between min. of TPE and min. of CE
   - $\mathrm{TPE}^* = -\mathrm{CE}^*$
   - existence and uniqueness of solution