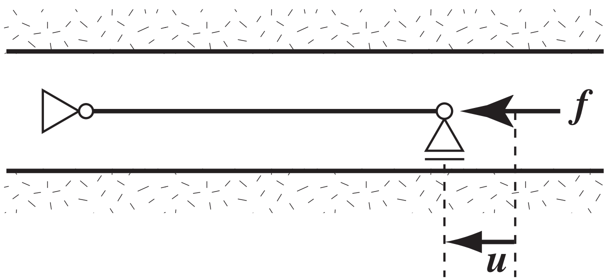
Arc-Length Method for Frictional Contact based on Mathematical Program with Complementarity Constraints

Yoshihiro Kanno[†] João A.C. Martins[‡]

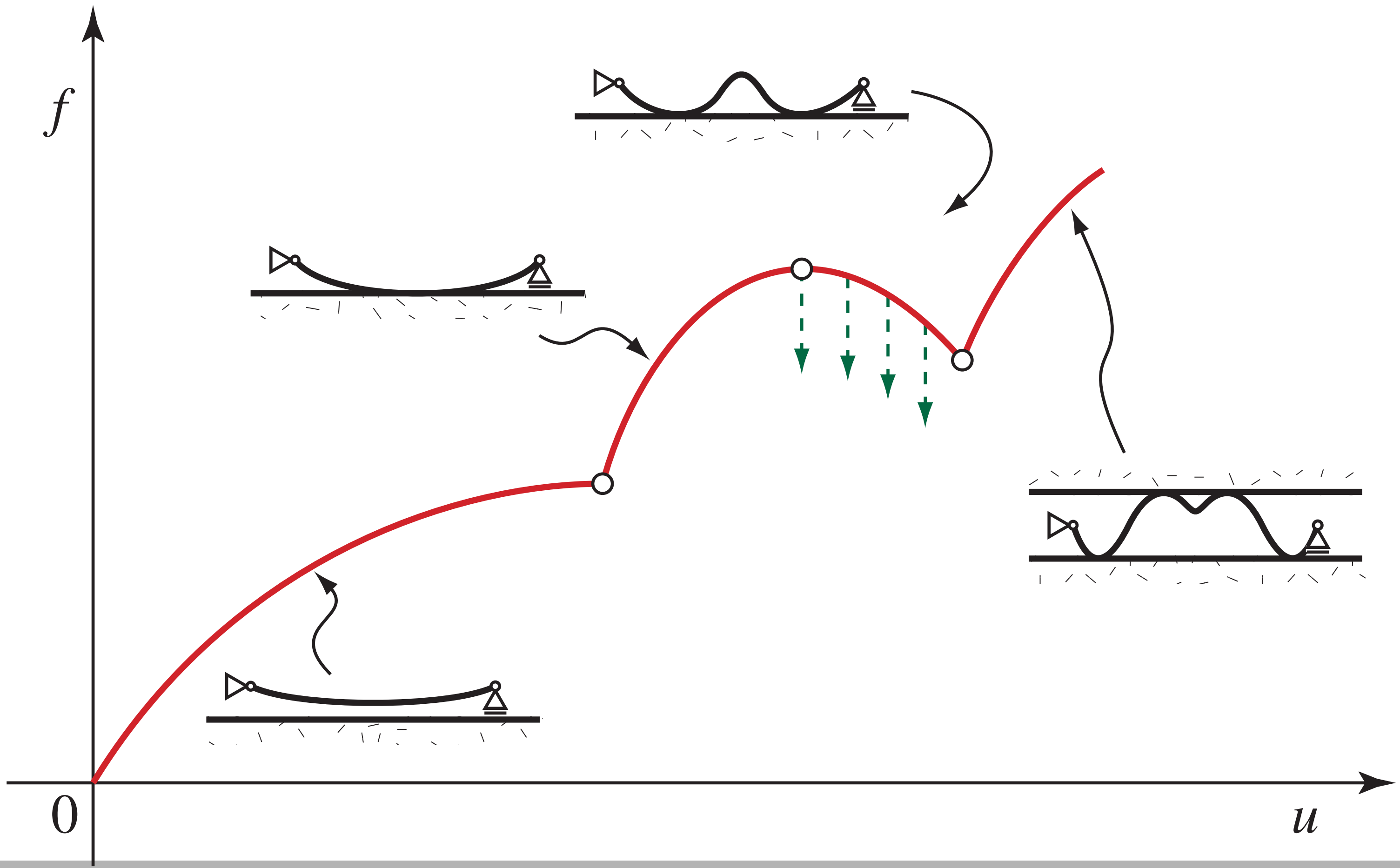
[†]Kyoto University (Japan)

[‡]Instituto Superior Técnico (Portugal)

Frictional contact problems



- contact / free
- slip / stick
- $\Longrightarrow$ limit / bifurcation points



Existence results of contact problems

- frictionless
  - continuation method [Miersemann & Mittelmann 89]
- frictional & large disp.
  - tangent stiffness [Wriggers 02]
  - arc-length method [Koo & Kwak 96]
  - approximation of friction law [Perí́c & Owen 92]
- selection of paths (small deformation)
  - min. of potential energy [Hilding 00]

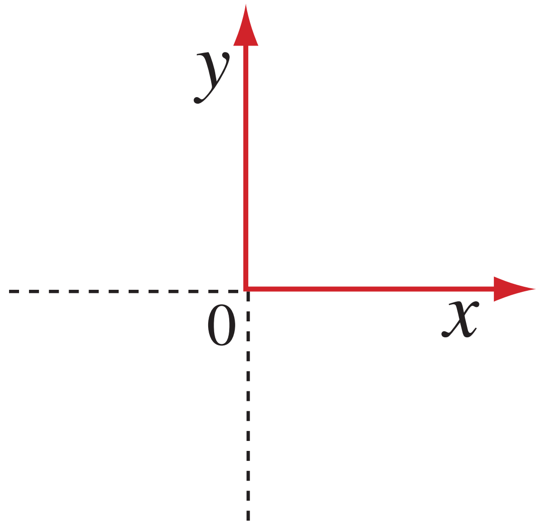
Existence results of contact problems

- frictionless
  - continuation method [Miersemann & Mittelmann 89]
- frictional & large disp.
  - tangent stiffness [Wriggers 02]
  - arc-length method [Koo & Kwak 96]
  - approximation of friction law [Períc & Owen 92]
- selection of paths (small deformation)
  - min. of potential energy [Hilding 00]

- $\Longrightarrow$ arc-length method $+$ path selection
- $\Longrightarrow$ solve an optimization problem at each increment

Complementarity condition

$$\boldsymbol{x} \geq \mathbf{0}, \quad \boldsymbol{y} \geq \mathbf{0}, \quad \boldsymbol{x}^\top \boldsymbol{y} = 0$$



Generally & for short, we often write

$$\mathbf{0} \leq (\boldsymbol{A}\boldsymbol{x}) \perp (\boldsymbol{B}\boldsymbol{y}) \geq \mathbf{0}$$

MPEC

- Mathematical Program with Complementarity Constraints = MPEC

$$\begin{aligned} \max \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0}, \\ & \mathbf{0} \leq \boldsymbol{h}(\boldsymbol{x}) \perp \boldsymbol{w}(\boldsymbol{x}) \geq \mathbf{0} \end{aligned}$$

complementarity condition:

$$\mathbf{0} \leq \boldsymbol{h}(\boldsymbol{x}) \perp \boldsymbol{w}(\boldsymbol{x}) \geq \mathbf{0}$$

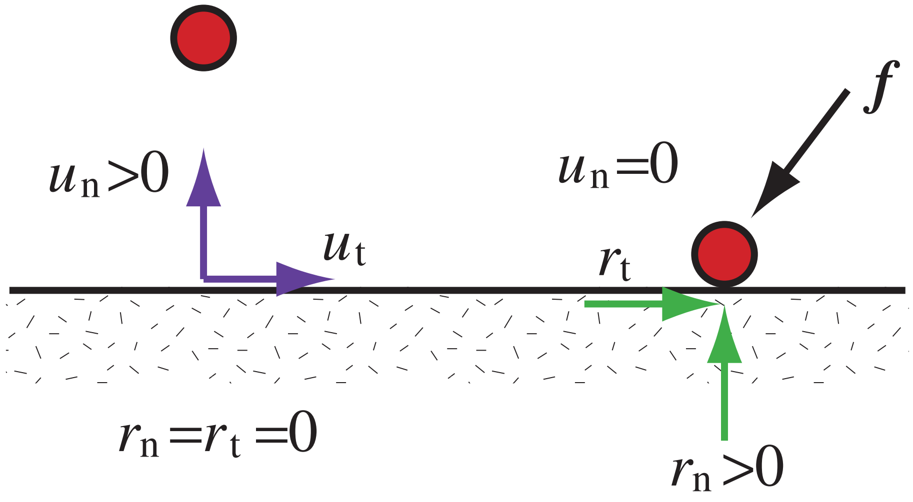
$$\Updownarrow$$

$$\boldsymbol{h}(\boldsymbol{x}) \geq \mathbf{0}, \quad \boldsymbol{w}(\boldsymbol{x}) \geq \mathbf{0}, \quad \boldsymbol{h}(\boldsymbol{x})^\top \boldsymbol{w}(\boldsymbol{x}) = 0$$

Contact problems

non-penetration condition :

$$u_{\rm n} > 0 \Longrightarrow r_{\rm n} = 0 \qquad : \text{free}$$
$$r_{\rm n} > 0 \Longrightarrow u_{\rm n} = 0 \qquad : \text{contact}$$

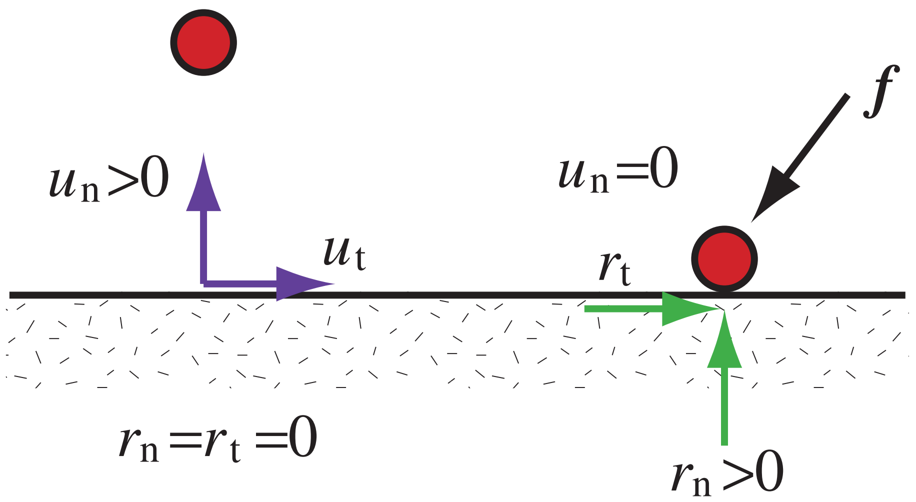


Contact problems

non-penetration condition :

$$u_n > 0 \implies r_n = 0 \qquad : \text{free}$$

$$r_n > 0 \implies u_n = 0 \qquad : \text{contact}$$



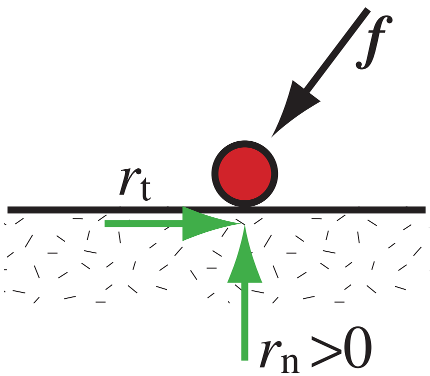
complementarity condition :

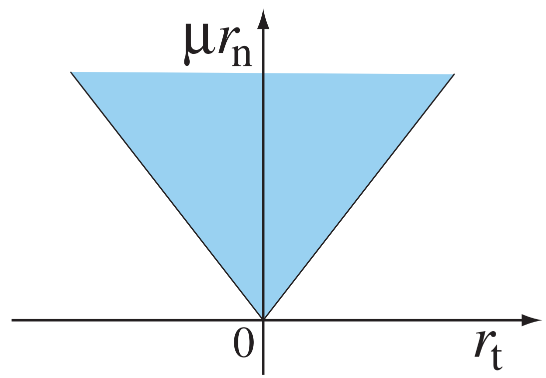
$$u_n \geq 0, \quad r_n \geq 0, \quad u_n r_n = 0.$$

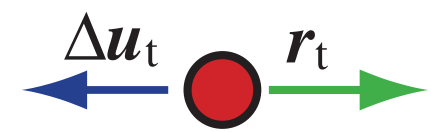
Coulomb's friction law

$$\mu r_{\mathrm{n}} \geq |r_{\mathrm{t}}|$$

$$\Delta \boldsymbol{u}_{\mathrm{t}} = -\alpha \boldsymbol{r}_{\mathrm{t}}, \quad \begin{cases} \alpha > 0 & \text{(slip)} \\ \alpha = 0 & \text{(stick)} \end{cases}$$







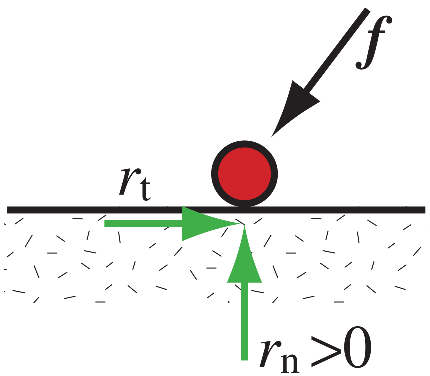
μ	: coefficient of friction
$(r_{\mathrm{n}}, r_{\mathrm{t}})$	: reaction (normal, tangential)

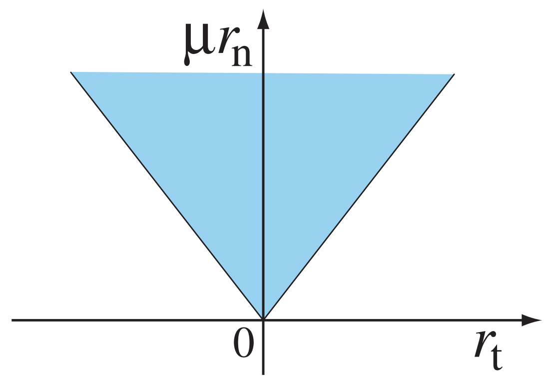
Coulomb's friction law

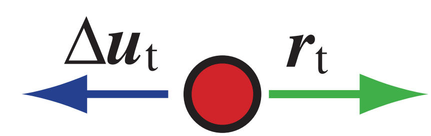
complementarity condition:

$$\mu r_{\mathrm{n}} \geq |r_{\mathrm{t}}|, \qquad \lambda_{\mathrm{n}} \geq |\Delta u_{\mathrm{t}}|$$

$$(\mu r_{\mathrm{n}}, r_{\mathrm{t}}) \cdot (\lambda_{\mathrm{n}}, \Delta u_{\mathrm{t}}) = 0$$



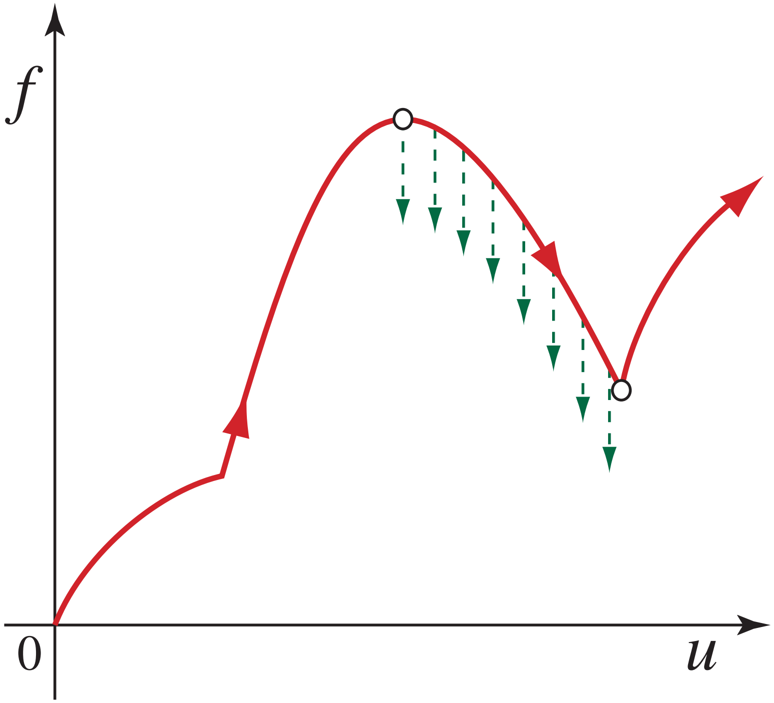




μ	: coefficient of friction
$(r_{\mathrm{n}}, r_{\mathrm{t}})$	: reaction (normal, tangential)

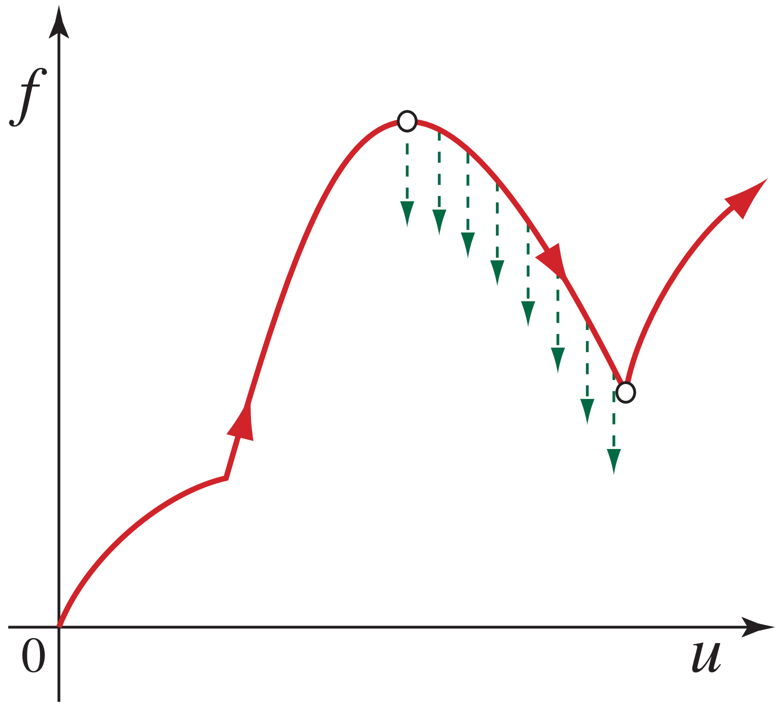
Aims of the talk

- quasi-static contact,
  Coulomb's friction
- limit points
  (smooth / nonsmooth)
- avoid bifurcated unloading
  paths



Aims of the talk

- quasi-static contact, Coulomb's friction
  - $\Longrightarrow$ complementarity conditions
- limit points (smooth / nonsmooth)
  - $\Longrightarrow$ arc-length method
- avoid bifurcated unloading paths
  - $\Longrightarrow$ path with maximum dissipation of energy



Arc-length method

At each increment,

- conventional arc-length method
  - $\boldsymbol{u}$: displacements
  - s : loading parameter
  - find $(\Delta \boldsymbol{u}^{(k)}, \Delta s^{(k)})$
    satisfying $\|(\Delta \boldsymbol{u}^{(k)}, \Delta s^{(k)})\| = \overline{S}$
    by solving nonlinear equations

- our method
  - find $(\Delta \boldsymbol{u}^{(k)}, \Delta s^{(k)})$ and $\boldsymbol{r}^{(k)}$
    satisfying $\|(\Delta \boldsymbol{u}^{(k)}, \Delta s^{(k)})\| = \overline{S}$
    by solving an MPEC

Choose the path with max. dissipation

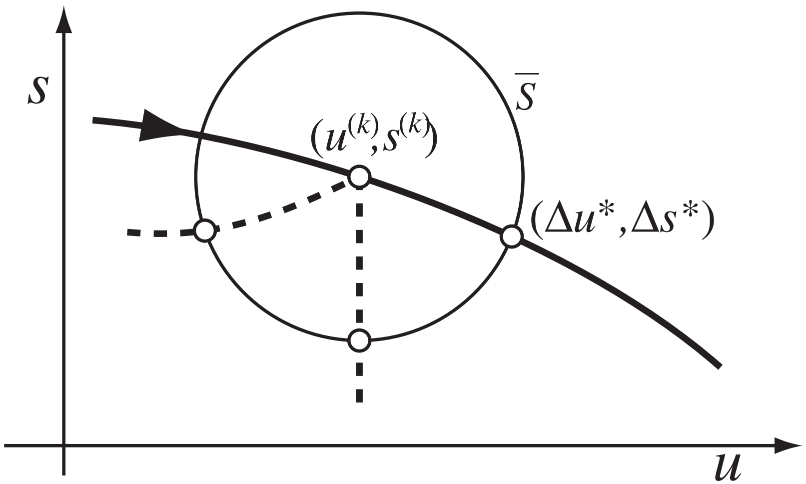
system of equilibrium paths (♣):

$$\phi(\Delta \boldsymbol{u}, \Delta s) - \boldsymbol{r} = \boldsymbol{0} \qquad \text{(equilibrium eqs.)}$$

$$\boldsymbol{0} \le \boldsymbol{A}\Delta \boldsymbol{u} \perp \boldsymbol{B}\boldsymbol{r} \ge \boldsymbol{0} \qquad \text{(non-penetration \& friction)}$$

hypersphere constraint on arc-length (♢):

$$\|(\Delta \boldsymbol{u}, \Delta s)\| = \overline{S}$$



Choose the path with max. dissipation

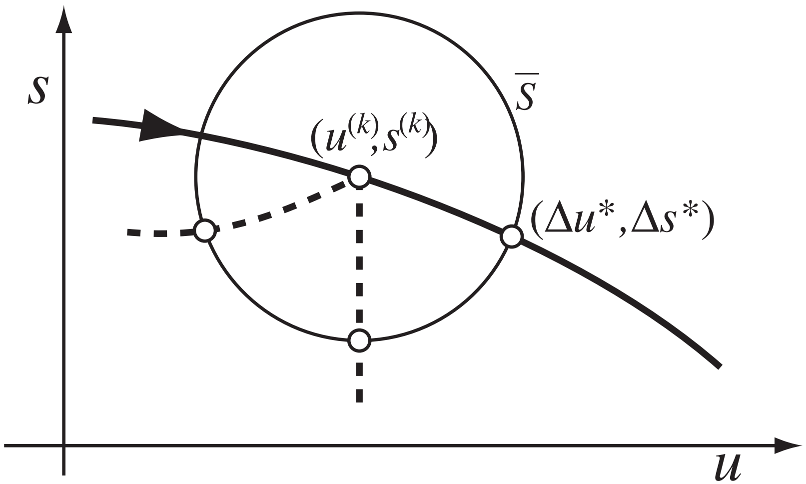
system of equilibrium paths (♣):

$$\boldsymbol{\phi}(\Delta \boldsymbol{u}, \Delta s) - \boldsymbol{r} = \boldsymbol{0} \qquad \text{(equilibrium eqs.)}$$

$$\boldsymbol{0} \leq \boldsymbol{A}\Delta \boldsymbol{u} \perp \boldsymbol{B}\boldsymbol{r} \geq \boldsymbol{0} \qquad \text{(non-penetration \& friction)}$$

hypersphere constraint on arc-length (♢):

$$\|(\Delta \boldsymbol{u}, \Delta s)\| = \overline{S}$$



prototype of incremental problem:

$$\begin{array}{ll} \max & -\boldsymbol{r}_{\mathrm{t}}^{(k)} \cdot \Delta \boldsymbol{u}_{\mathrm{t}} \qquad \text{(dissipation of energy)} \\ \text{s.t.} & (♣) \;\&\; (♢) \end{array}$$

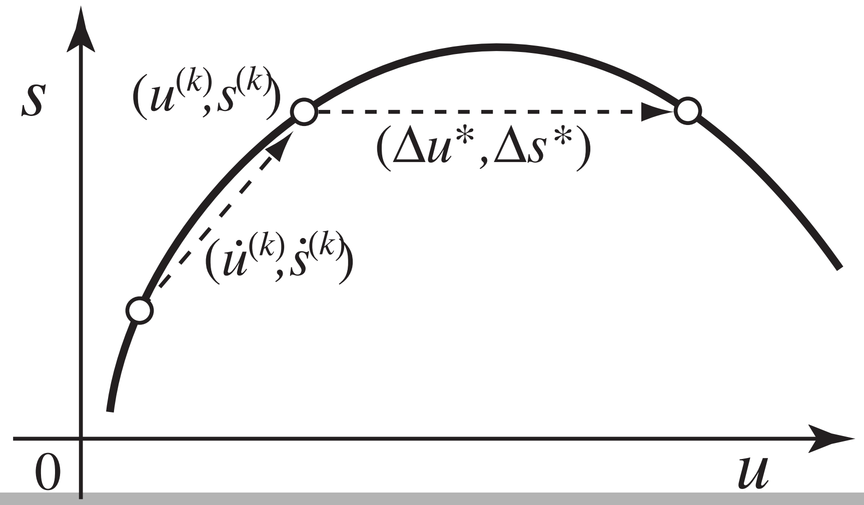
MPEC

$$\begin{aligned} \max \quad & (\text{dissipation}) - p(\Delta \boldsymbol{u}, \Delta s) \\ \text{s.t.} \quad & (\text{equilibrium eqs.}), \\ & (\text{complementarity conds.}) \end{aligned}$$

- penalty

$p = \frac{1}{2}\gamma_1^k \|(\Delta \boldsymbol{u}, \Delta s)\|^2$ $\Longleftarrow$ on arc-length

$- \gamma_2^k (\dot{\boldsymbol{u}}^{(k)}, \dot{s}^{(k)}) \cdot (\Delta \boldsymbol{u}, \Delta s)$ $\Longleftarrow$ prevents ‘backward’ solution



MPEC

$$\begin{aligned} \max \quad & (\text{dissipation}) - p(\Delta \boldsymbol{u}, \Delta s) \\ \text{s.t.} \quad & (\text{equilibrium eqs.}), \\ & (\text{complementarity conds.}) \end{aligned}$$

- penalty

  $p = \frac{1}{2}\gamma_1^k \|(\Delta \boldsymbol{u}, \Delta s)\|^2$ $\Longleftarrow$ on arc-length

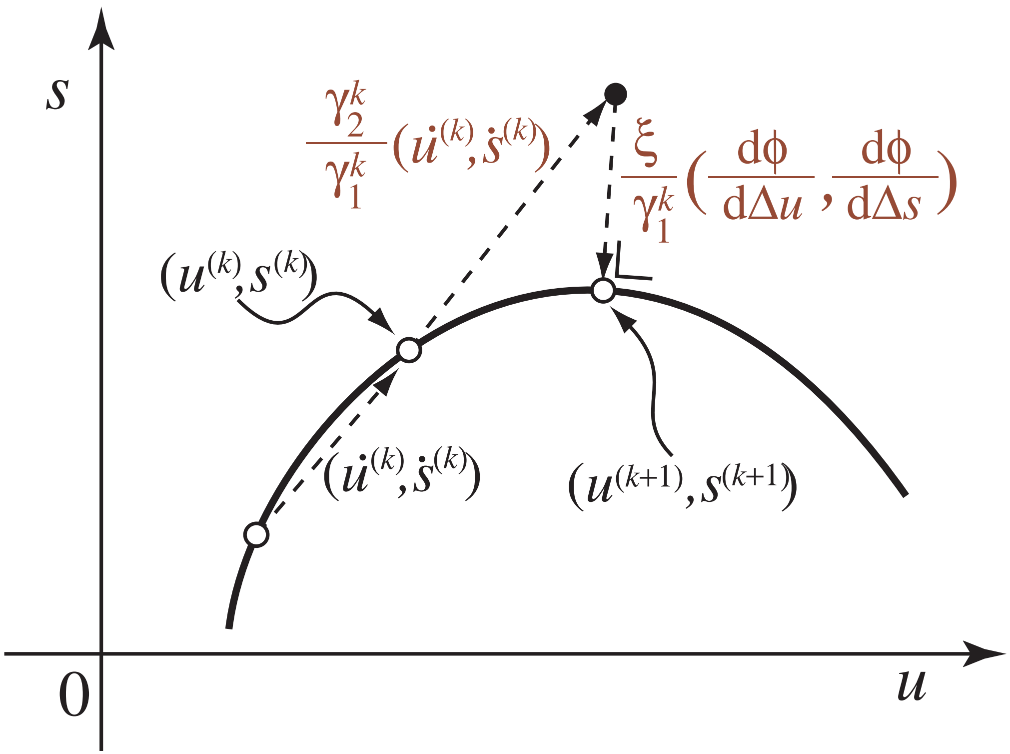
  $- \gamma_2^k (\dot{\boldsymbol{u}}^{(k)}, \dot{s}^{(k)}) \cdot (\Delta \boldsymbol{u}, \Delta s)$ $\Longleftarrow$ prevents ‘backward’ solution

- MPEC has
  - convex objective function
  - feasible solutions
- control $\|(\Delta \boldsymbol{u}, \Delta s)\|$ by choosing γ_1^k

Characterization of solution ($r_t^{(k)} = 0$)

KKT conditions:

$$\begin{pmatrix} \boldsymbol{u}^{(k+1)} \\ s^{(k+1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}^{(k)} \\ s^{(k)} \end{pmatrix} + \frac{\gamma_2^k}{\gamma_1^k} \begin{pmatrix} \dot{\boldsymbol{u}}^{(k)} \\ \dot{s}^{(k)} \end{pmatrix} + \frac{1}{\gamma_1^k} \sum_{l=1}^{n^{\mathrm{d}}} \xi_l \begin{pmatrix} \partial\phi_l / \partial\Delta\boldsymbol{u} \\ \partial\phi_l / \partial\Delta s \end{pmatrix}$$



Regularization of MPEC

regularization:

$$\mu r_{\mathrm{n}} \geq |r_{\mathrm{t}}|, \quad \lambda_{\mathrm{n}} \geq |\Delta u_{\mathrm{t}}|$$

$$(\mu r_{\mathrm{n}}, r_{\mathrm{t}}) \cdot (\lambda_{\mathrm{n}}, \Delta u_{\mathrm{t}}) = 0$$

$$\Downarrow$$

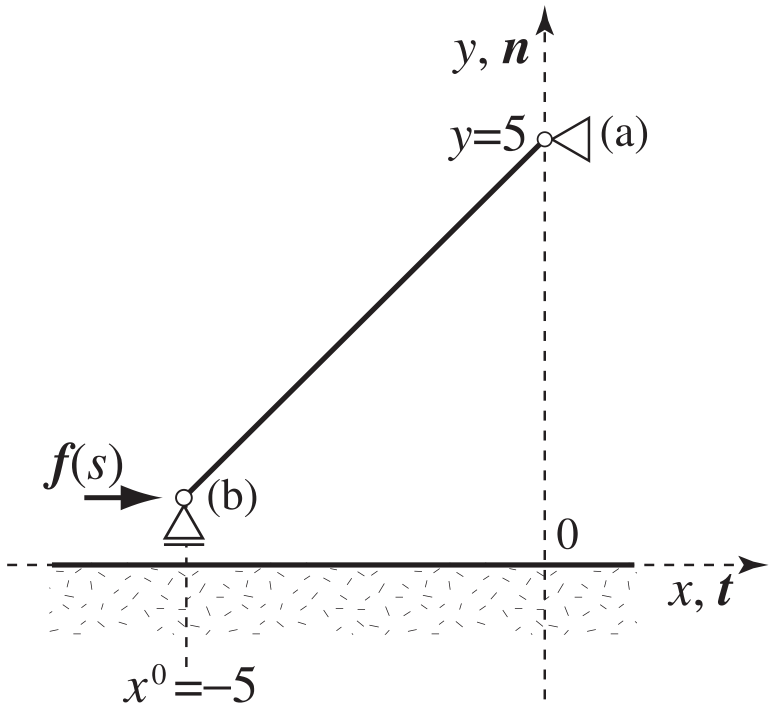
$$\mu r_{\mathrm{n}} \geq |r_{\mathrm{t}}|, \quad \lambda_{\mathrm{n}} \geq |\Delta u_{\mathrm{t}}|$$

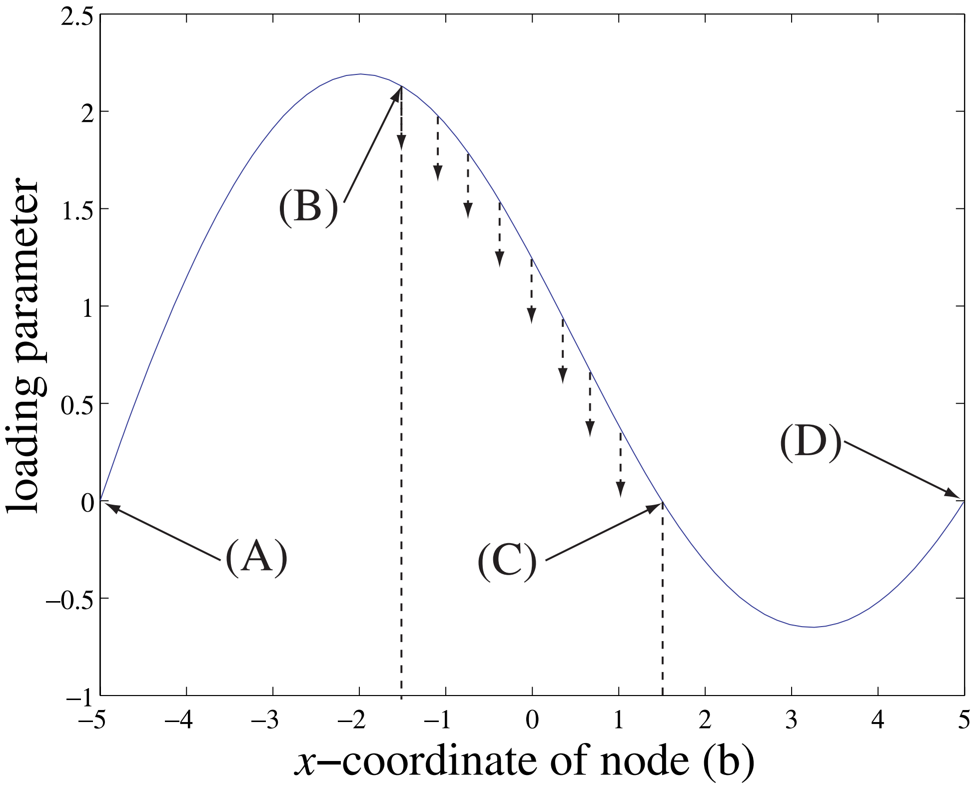
$$(\mu r_{\mathrm{n}}, r_{\mathrm{t}}) \cdot (\lambda_{\mathrm{n}}, \Delta u_{\mathrm{t}}) \leq \epsilon, \quad \epsilon > 0$$

- solve the sequence of regularized problems
  - by decreasing $\epsilon \downarrow 0$
  - by using conventional SQP
    - Matlab Optimization Toolbox (Ver. 2.1)

One-bar truss

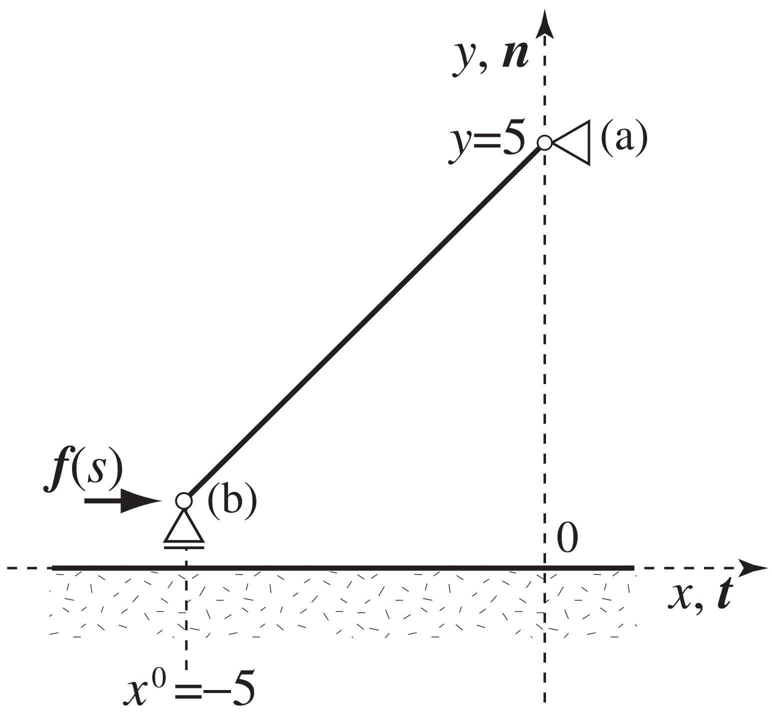
- analogous to [Mróz 02] ($\mu = 0.3$)
- smooth limit points
- (B)→(C) : bifurcation points

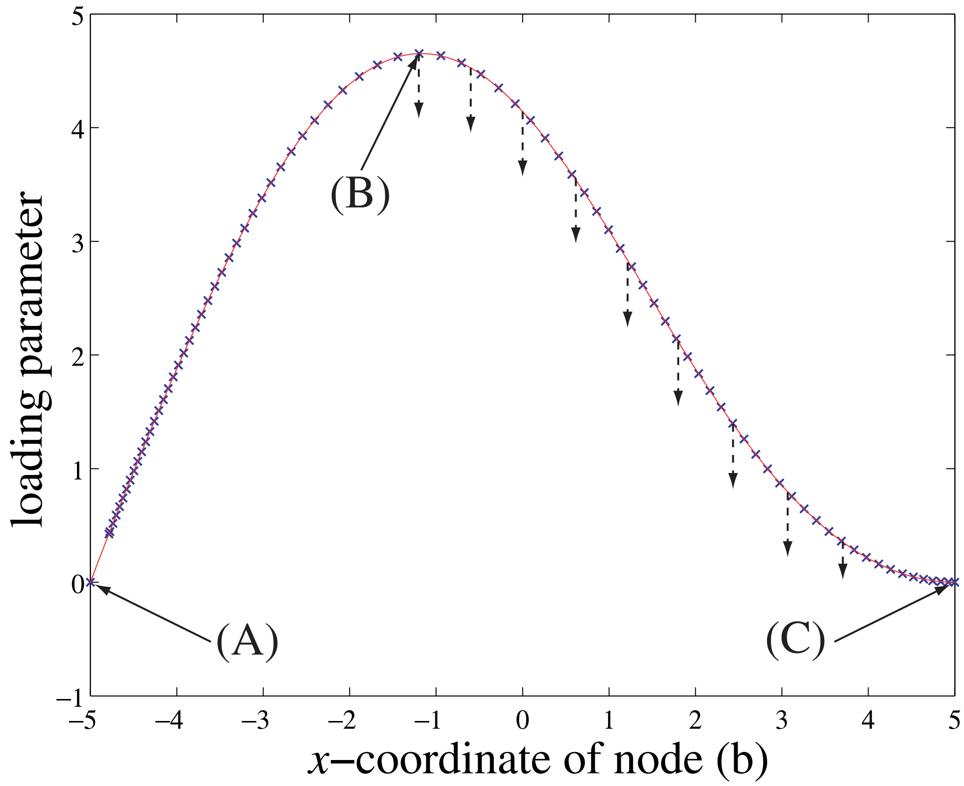




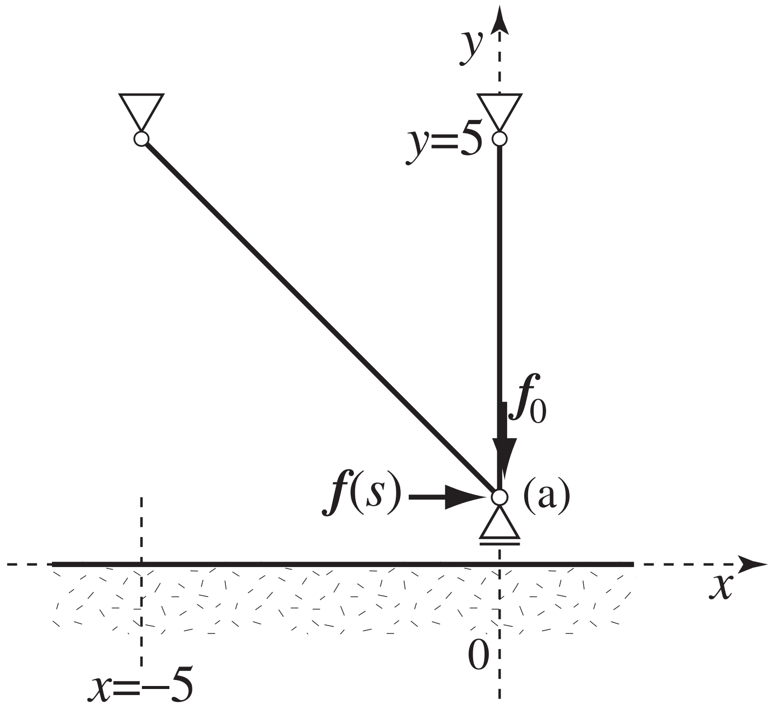
One-bar truss

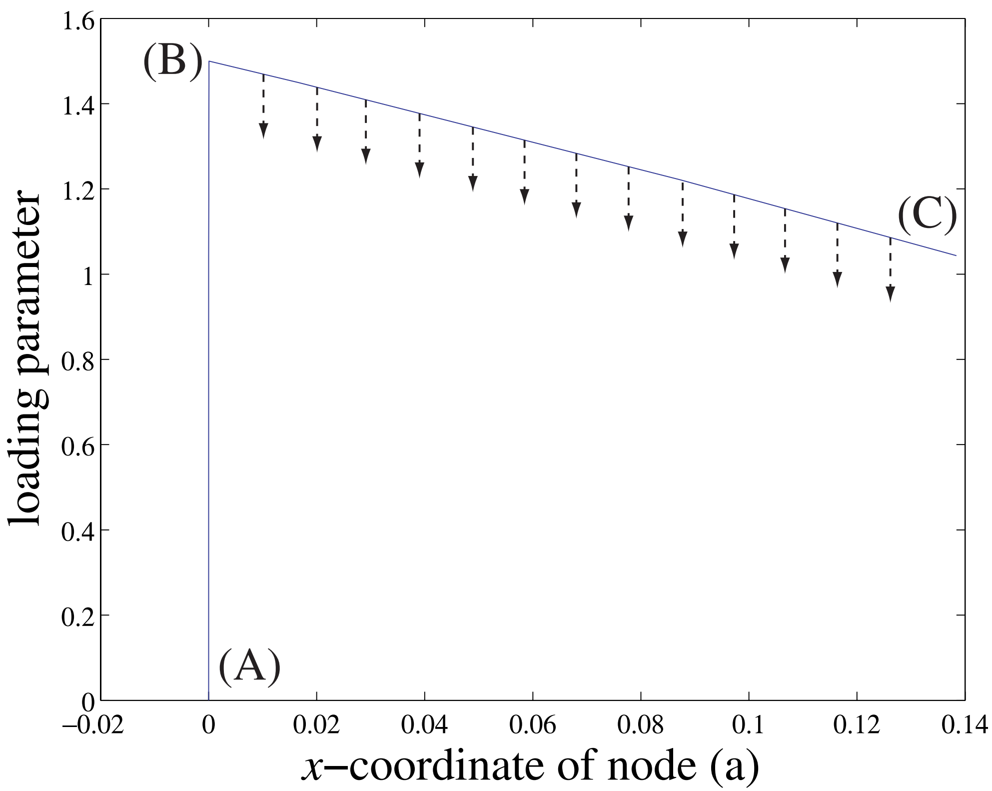
- analogous to [Mróz 02] ($\mu = 1.0$)
- – : analytical, $\times$: our method
- (B)$\rightarrow$(C) : bifurcation points





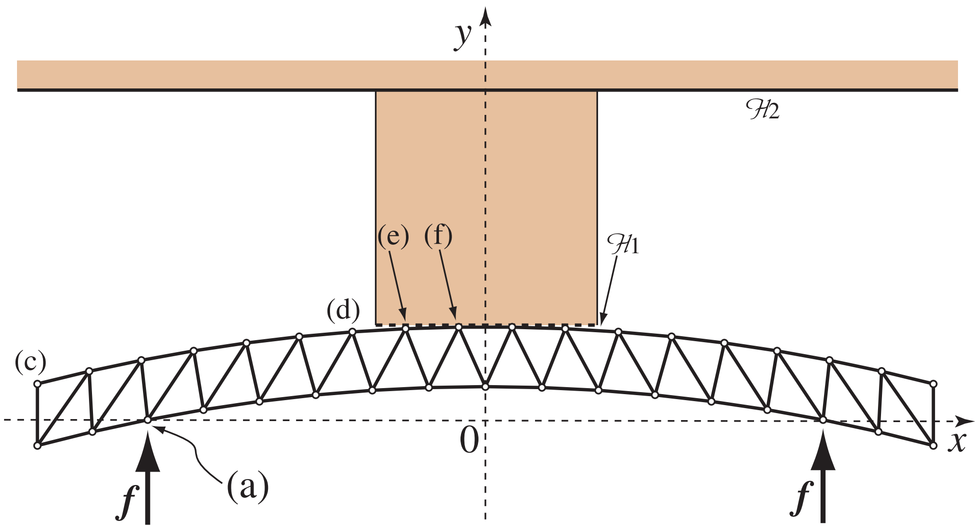
Ex. of [Klarbring 90]

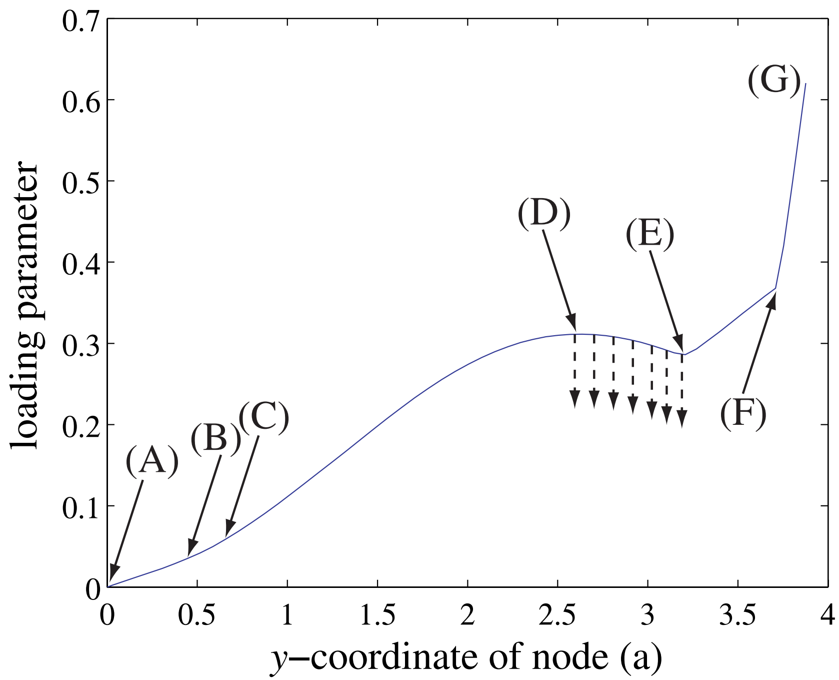




- $\mu = 1.5$
- (B) : nonsmooth limit point
- (B)→(C) : bifurcation points

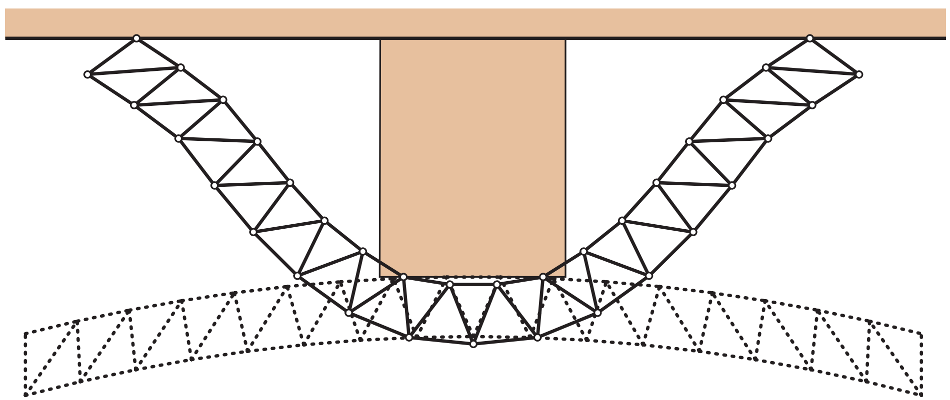
Arch-type truss

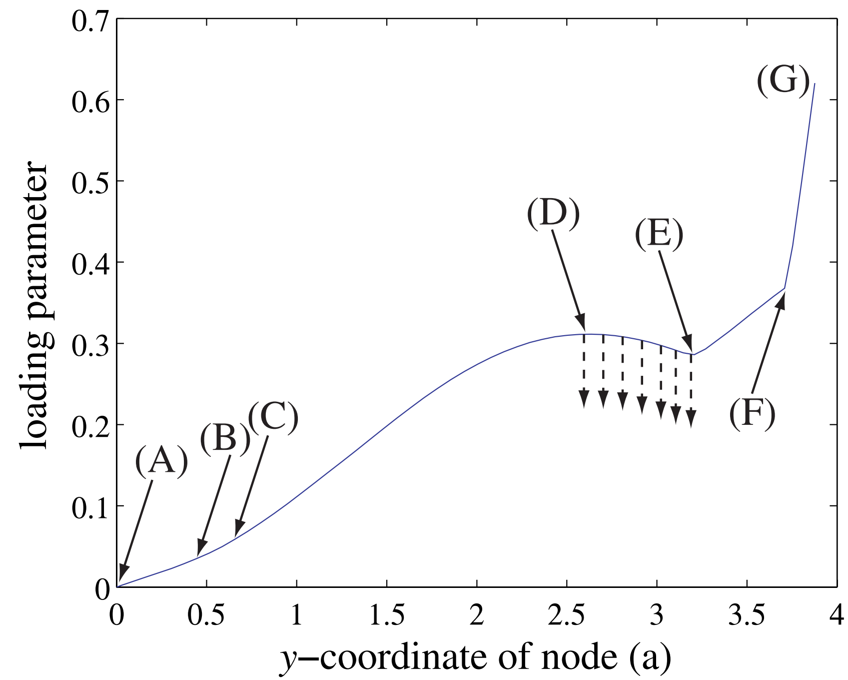




- (D) : smooth limit point
- (E) : nonsmooth limit point
- (D)$\longrightarrow$(E) : bifurcation points

Arch-type truss





- (D) : smooth limit point
- (E) : nonsmooth limit point
- (D)→(E) : bifurcation points

Conclusion

- Arc-length method
  - with
    - contact, the Coulomb friction
    - choosing the path with the max. dissipation
    - geometrical nonlinearity
- MPEC is solved
  - at each increment
  - by using regularization and SQP method
- Trace equilibrium paths with
  - smooth / nonsmooth limit points
  - successive bifurcation points