Ellipsoidal Bounds for Static Response of Uncertain Trusses by using Semidefinite Programming

Yoshihiro Kanno[†] Izuru Takewaki[‡]

[†]University of Tokyo (Japan)

[‡]Kyoto University (Japan)

uncertainty

- stochastic model
  - reliability design
- non-stochastic model
  - unknown-but-bounded

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  - convex model [Ben-Haim & Elishakoff 90]
  - interval analysis [Alefeld & Mayer 00], [Chen *et al.* 02], etc.
  - mathematical programming [Calafiore & El Ghaoui 04]

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- stochastic model
  - reliability design
- non-stochastic model
  - unknown-but-bounded
  - convex model [Ben-Haim & Elishakoff 90]
    - linear approximation $\longrightarrow$ small perturbation
  - interval analysis [Alefeld & Mayer 00], [Chen *et al.* 02], etc.
    - conservative $\longrightarrow$ often too conservative
  - mathematical programming [Calafiore & El Ghaoui 04]

uncertain equilibrium eqtn.

$$\boldsymbol{K}(\mathsf{a})\boldsymbol{u} = \mathsf{f}$$

uncertainty :

$$\mathsf{a}_i = \widetilde{a}_i + a_i^0 \zeta_{\mathrm{a}i}, \quad \zeta_{\mathrm{a}} \in \mathcal{Z}_{\mathrm{a}}$$

$$\mathsf{f}_j = \widetilde{f}_j + f_j^0 \zeta_{\mathrm{f}j}, \quad \zeta_{\mathrm{f}} \in \mathcal{Z}_{\mathrm{f}}$$

$\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{f}}$	nominal
$\zeta_{\mathrm{a}}, \zeta_{\mathrm{f}}$	unknown-but-bounded
$\boldsymbol{a}^0, \boldsymbol{f}^0 > \boldsymbol{0}$	coefficients ('magnitudes' of uncertainties)
$\mathcal{Z}_{\mathrm{a}}, \mathcal{Z}_{\mathrm{f}}$	(given) closed sets

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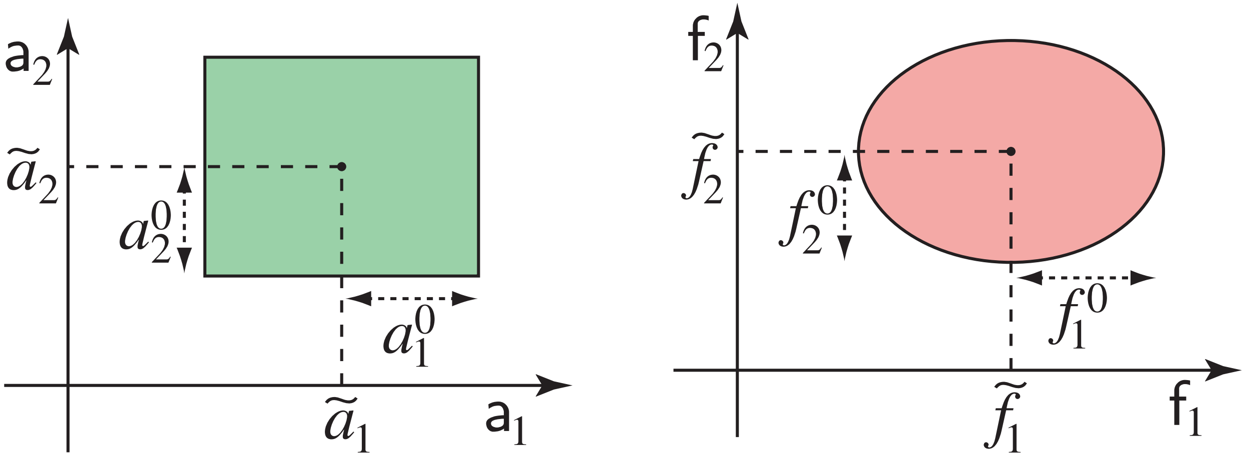
$$\mathcal{Z}_{\mathrm{f}} = \{\boldsymbol{\zeta}_{\mathrm{f}} : 1 \geq \|\boldsymbol{T}_l \boldsymbol{\zeta}_{\mathrm{f}}\|_2\,,\ \forall l\}$$



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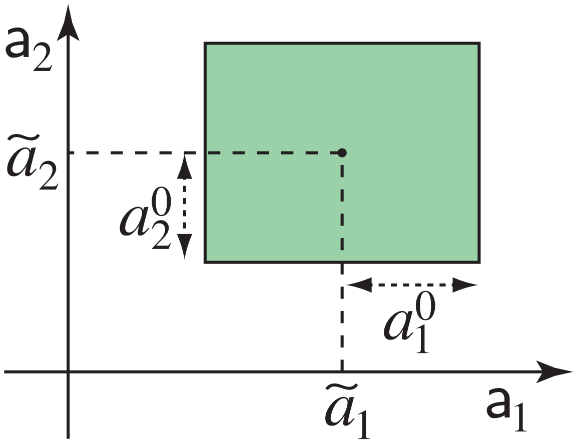
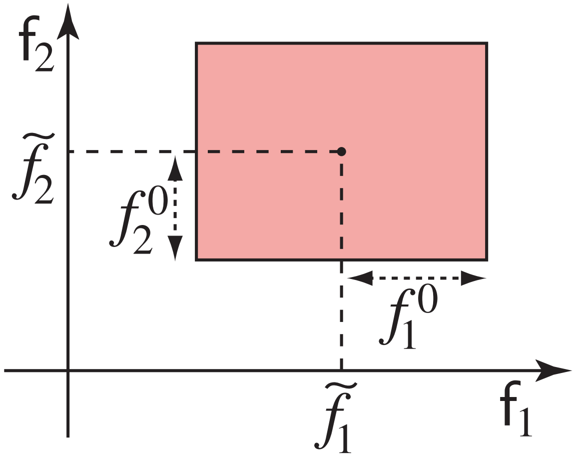




≡“interval analysis”

bounding ellipsoid

n-dimensional ellipsoid

$$\boldsymbol{x} = \hat{\boldsymbol{x}} + \boldsymbol{D}\boldsymbol{z}, \quad 1 \geq \|\boldsymbol{z}\|_2$$

$\hat{\boldsymbol{x}}$	: center
$\boldsymbol{P} := \boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}$	: symmetry, positive semidefinite

$$\mathcal{E}(\boldsymbol{P}, \hat{\boldsymbol{x}}) = \left\{ \boldsymbol{x} \in \mathbf{R}^n : \begin{pmatrix} \boldsymbol{P} & (\boldsymbol{x} - \hat{\boldsymbol{x}}) \\ (\boldsymbol{x} - \hat{\boldsymbol{x}})^{\mathrm{T}} & 1 \end{pmatrix} \text{ is } \textit{p.s.d.} \right\}$$

'size' of $\mathcal{E}(\boldsymbol{P}, \hat{\boldsymbol{x}})$

$$\mathrm{tr}(\boldsymbol{P}) \quad : \sum (\text{semi-axis length})^2$$

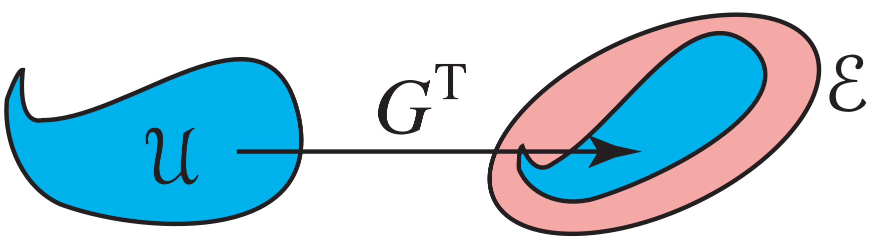
min. bounding ellipsoid

def. of ‘bounding ellipsoid’

$$\mathcal{E}(\boldsymbol{P}, \hat{\boldsymbol{x}}) \supseteq \{\boldsymbol{G}^{\mathrm{T}}\boldsymbol{u} | \, \boldsymbol{u} \in \mathcal{U}\} \qquad (\clubsuit)$$

$\mathcal{U}$: set of $\boldsymbol{u}$ solving uncertain equilibrium eqtn.

$\boldsymbol{G}^{\mathrm{T}}\boldsymbol{u}$: static response (to be estimated)



to find ‘tight’ bound

$$\min_{\boldsymbol{P}, \hat{\boldsymbol{x}}} \{\mathrm{tr}(\boldsymbol{P}) : (\clubsuit)\}$$

$\mathcal{S}$-lemma

$f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x}), g(\boldsymbol{x})$: quadratic functions

(a): $f_1(\boldsymbol{x}) \geq 0, \dots, f_m(\boldsymbol{x}) \geq 0 \quad \Longrightarrow \quad g(\boldsymbol{x}) \geq 0$

$\Uparrow$

(b): $\exists \boldsymbol{\rho} \geq \mathbf{0}, \quad g(\boldsymbol{x}) - \sum_{i=1}^{m} \rho_i f_i(\boldsymbol{x}) \geq 0, \ \forall \boldsymbol{x}$

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- in the form of Farkas lemma

(a)$'$: the system $f_1(\boldsymbol{x}) \geq 0, \dots, f_m(\boldsymbol{x}) \geq 0, \ g(\boldsymbol{x}) < 0$
is not solvable

SDP formulation

$$\mathcal{E}(\boldsymbol{P}, \hat{\boldsymbol{x}}) \supseteq \{\boldsymbol{G}^{\mathrm{T}}\boldsymbol{u} | \, \boldsymbol{u} \in \mathcal{U}\}$$

$\Updownarrow$ quadratic embedding

quadratic inequalities

$\Uparrow$ $\mathcal{S}$-lemma

p.s.d. constraint

SDP formulation

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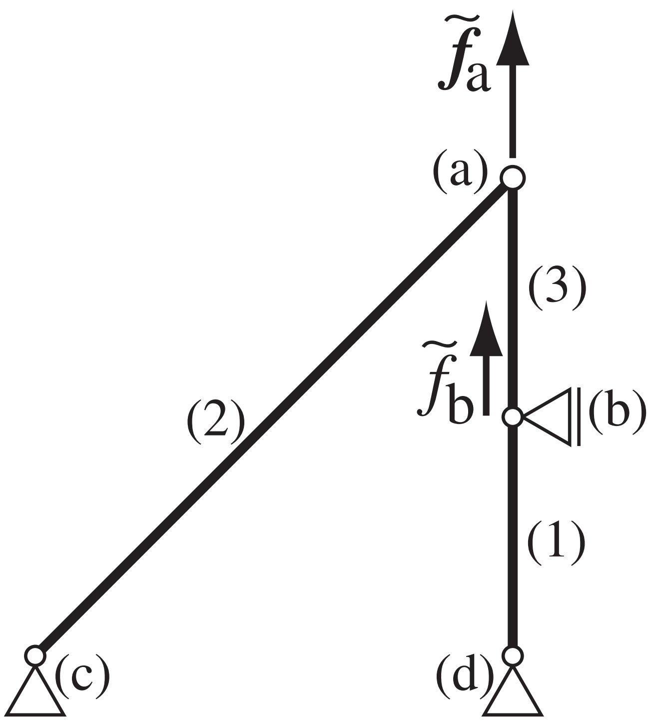
semidefinite programming (SDP) relaxation

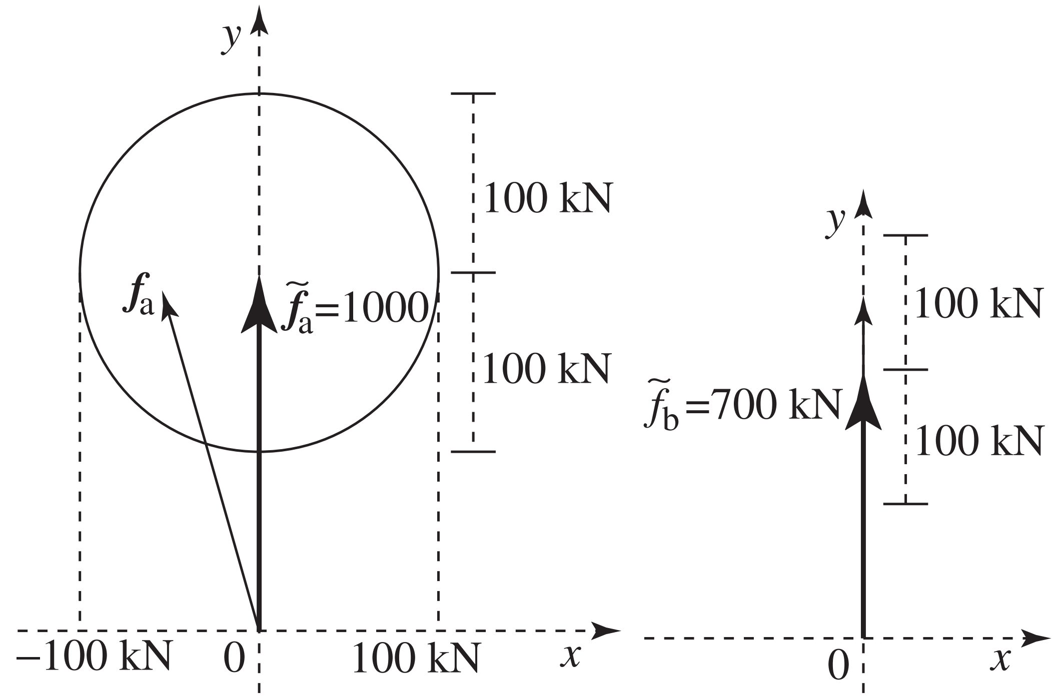
$$\min_{\boldsymbol{P}, \hat{\boldsymbol{x}}, \boldsymbol{\rho}} \{\mathrm{tr}(\boldsymbol{P}) : \boldsymbol{H}(\boldsymbol{P}, \hat{\boldsymbol{x}}, \boldsymbol{\rho}) \text{ is } \textit{p.s.d.}, \ \boldsymbol{\rho} \geq 0\}$$

- convex optimization
- can be solved effectively (in polynomial time)
  - primal-dual interior-point method

example (3-bar truss)

- uncertain stiffnesses
- uncertain loads





$$27 \leq a_1 \leq 33 \text{ cm}^2$$

$$47 \leq a_1 \leq 53 \text{ cm}^2$$

$$17 \leq a_1 \leq 23 \text{ cm}^2$$

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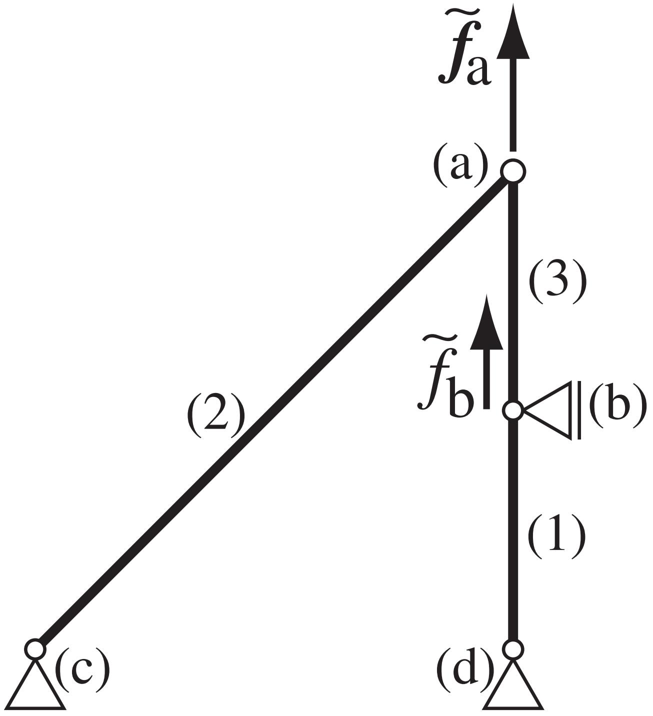

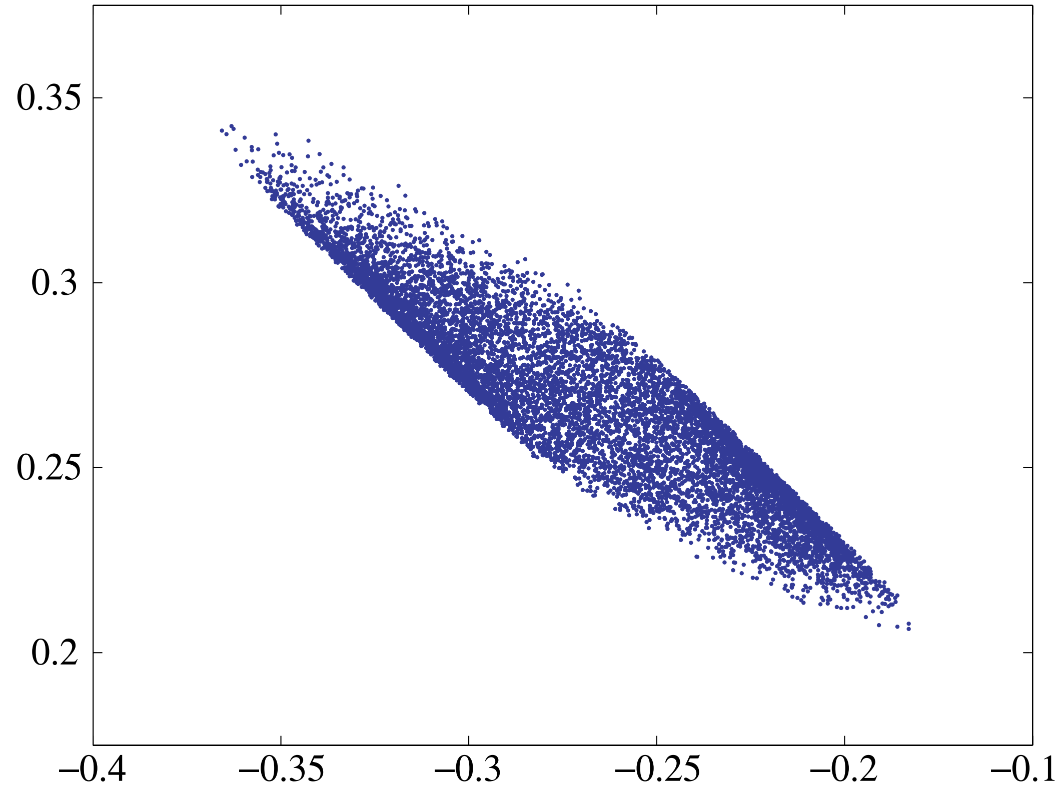

distribution of $\boldsymbol{u}_{\mathrm{a}}$

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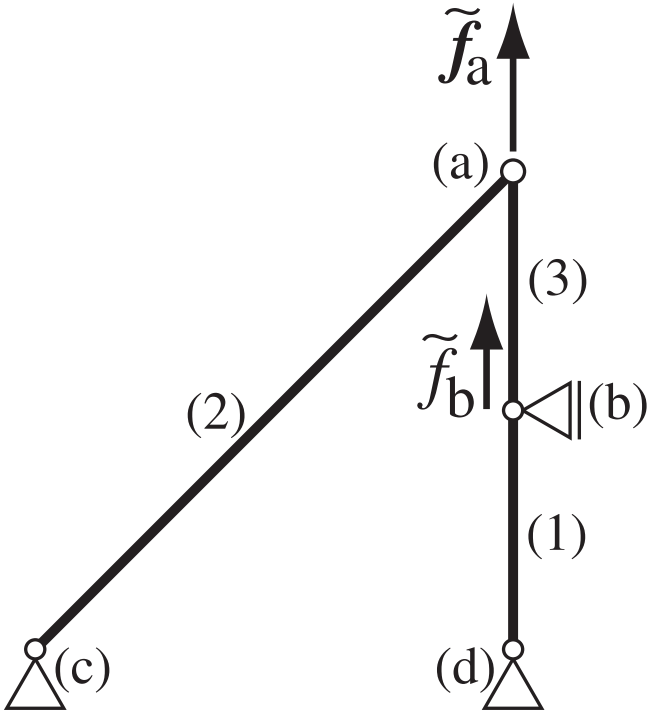

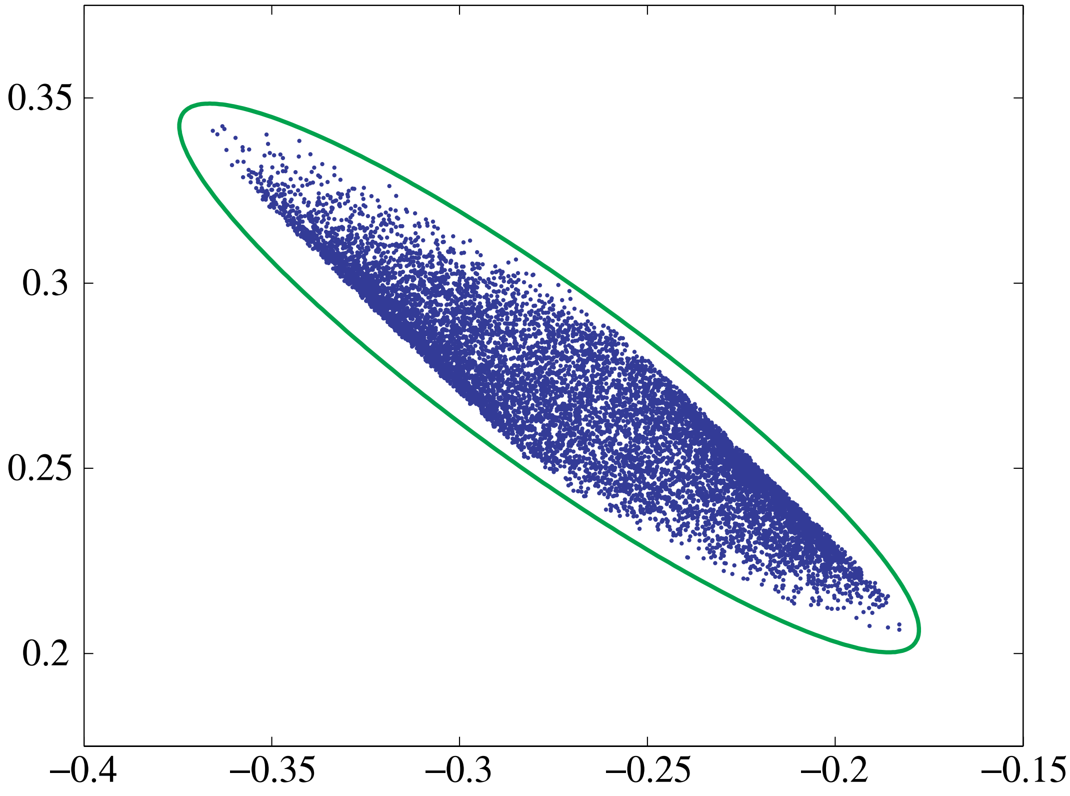

bounding ellipsoid for $\boldsymbol{u}_{\mathrm{a}}$

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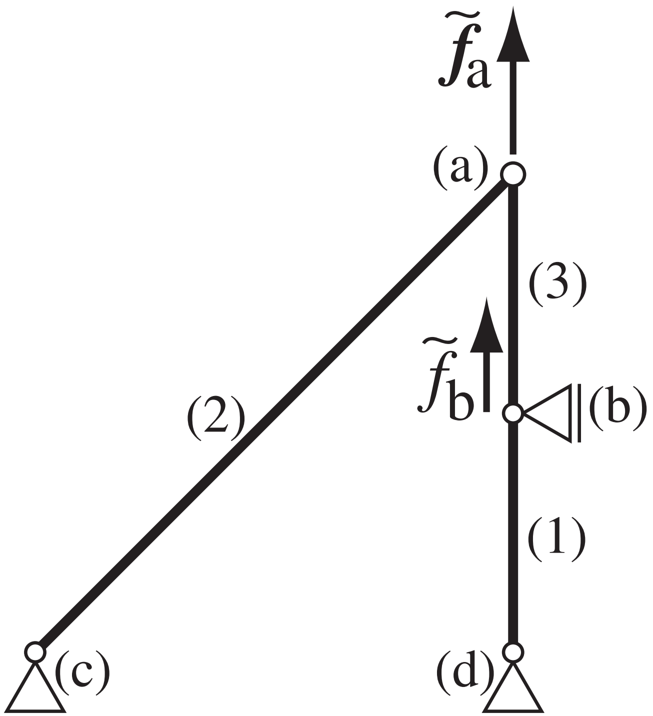
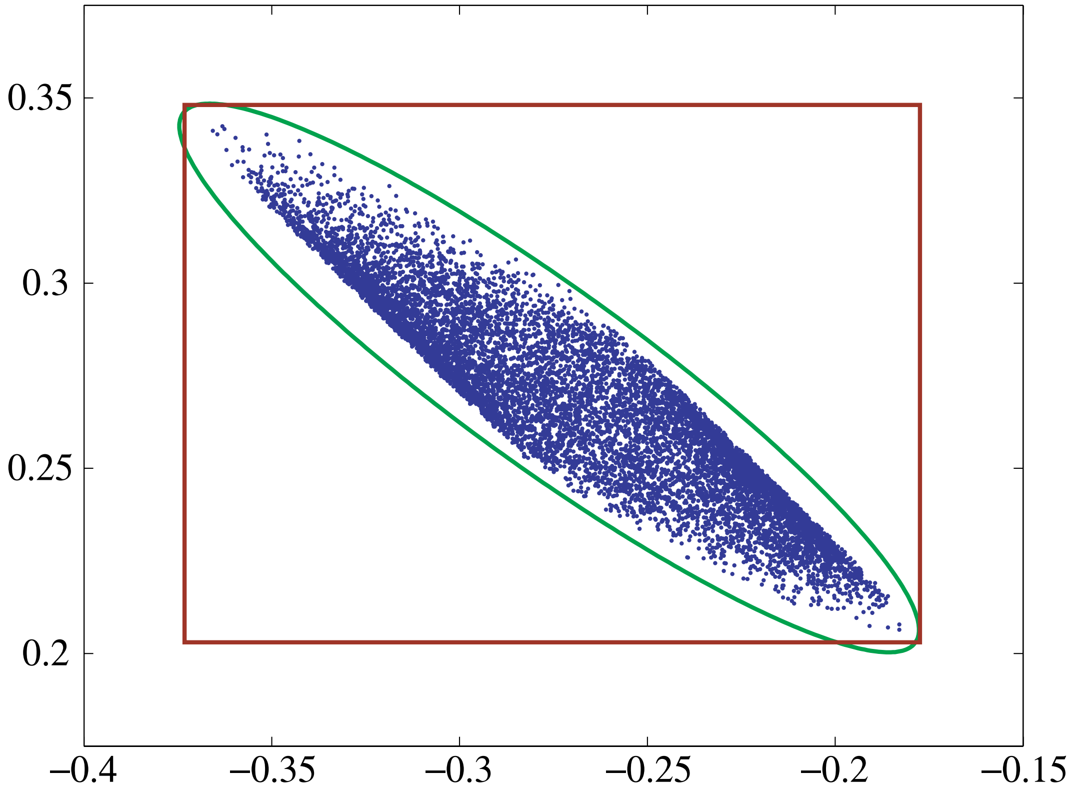
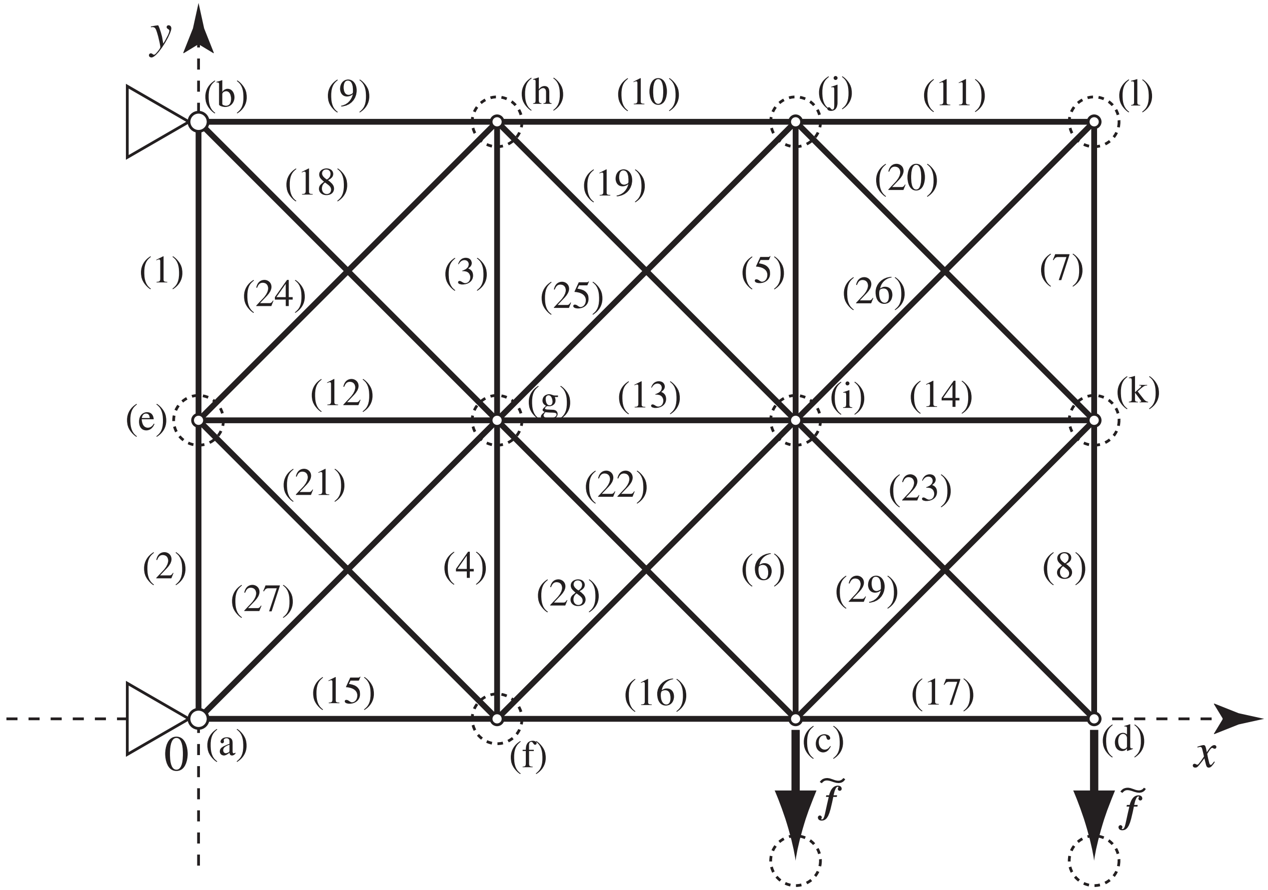




bounding box for $\boldsymbol{u}_a$

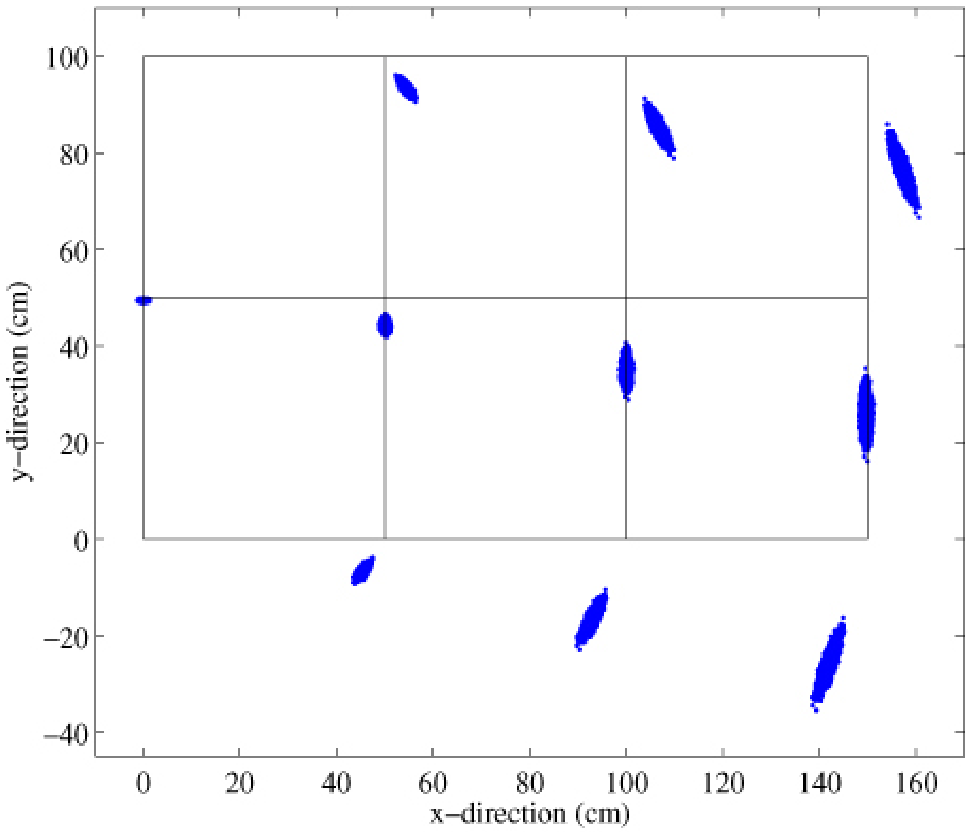
example (29-bar truss)



- nominal load : $\tilde{f} = 1000 \text{ kN}$
- uncertain load : $\|f^0\zeta_{\mathrm{f}}\| \leq 100 \text{ kN}$, $\forall$ node
- uncertainty of stiffness : $9.0 \leq a_i \leq 11.0 \text{ cm}^2$

example (29-bar truss)

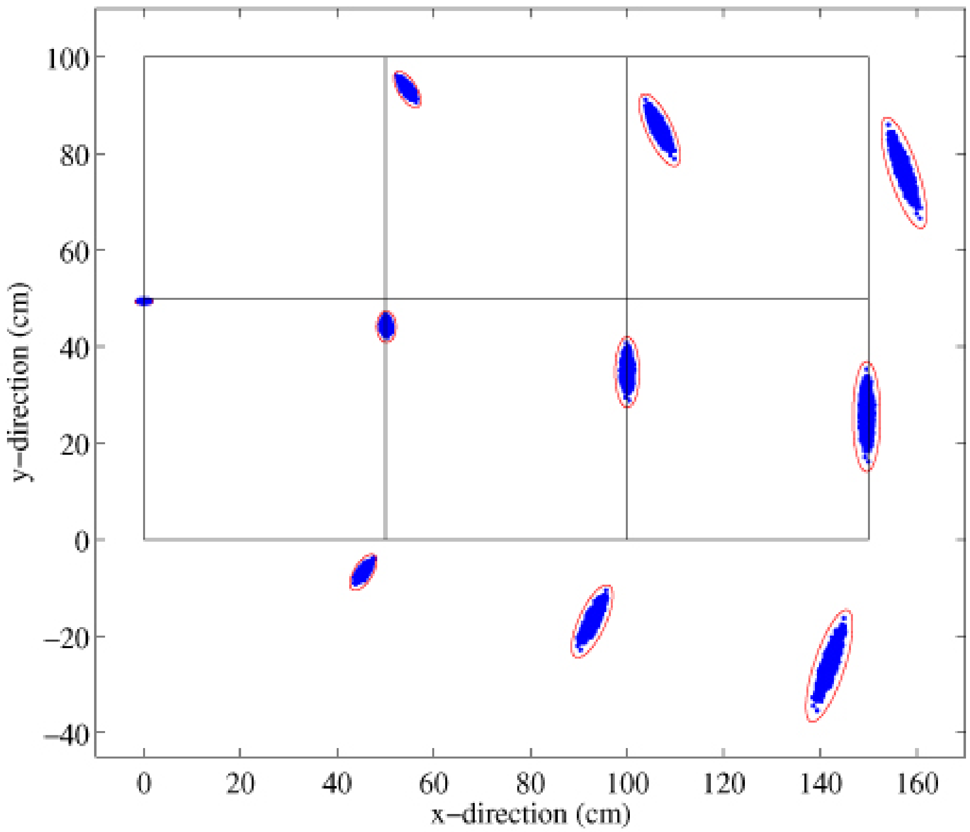
- distribution of u



(displacements $\times 10$)

example (29-bar truss)

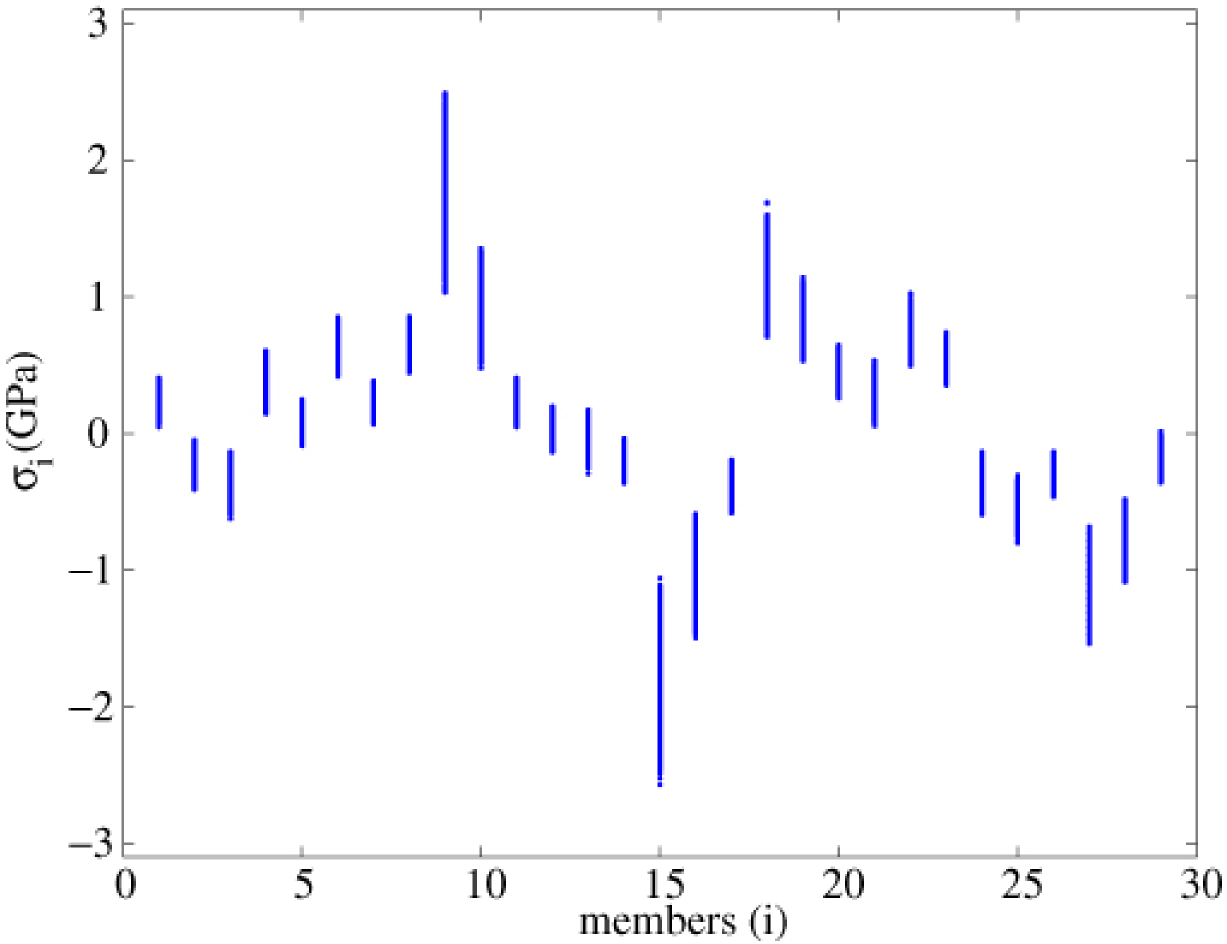
- distribution of u — bounding ellipsoids



(displacements $\times 10$)

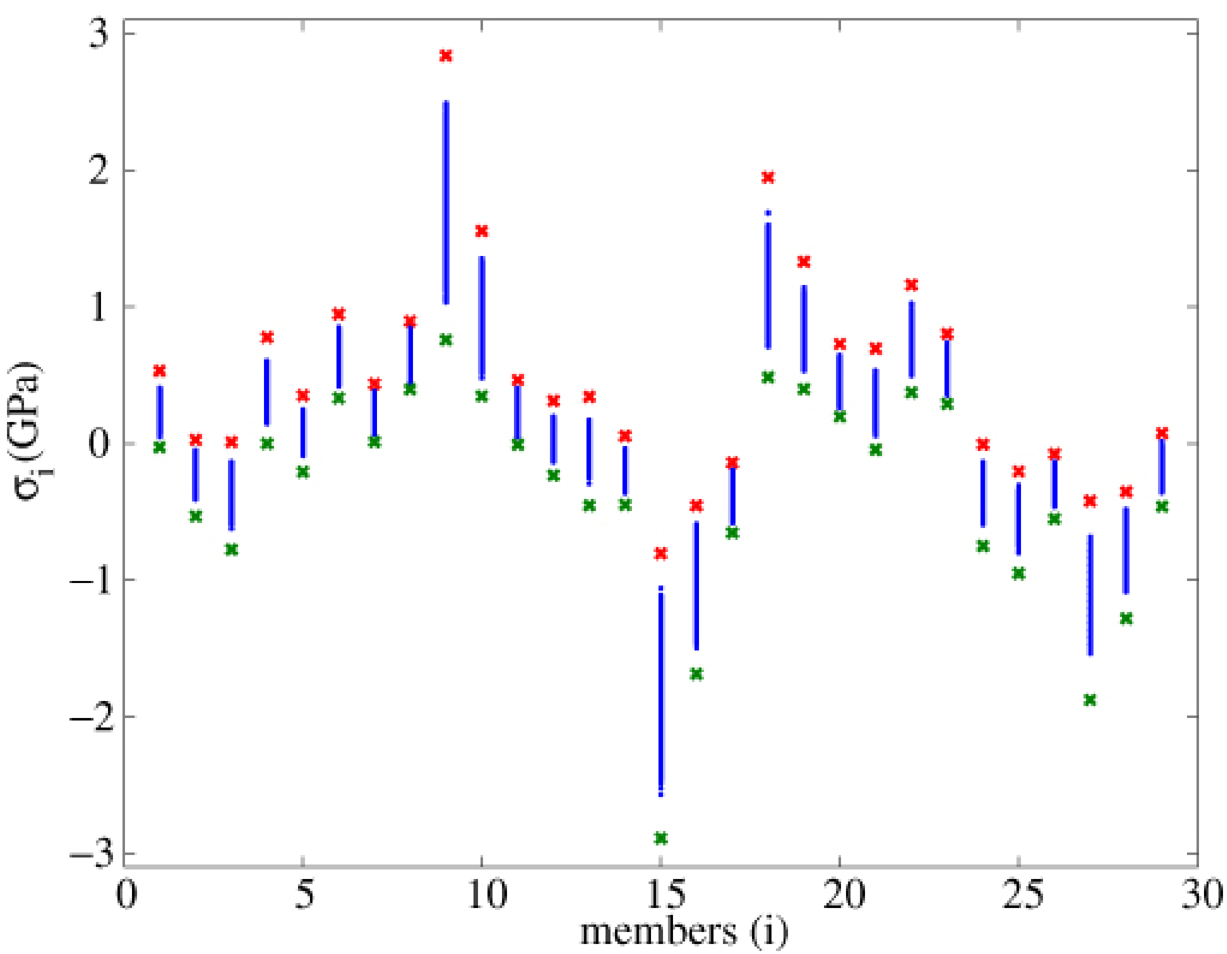
example (29-bar truss)

- distribution of stresses



example (29-bar truss)

- distribution of stresses — bounding intervals



conclusions

- uncertainty of trusses
  - uncertain stiffness & uncertain load
  - non-stochastic uncertainty
  - bounding ellipsoid of static response
- ellipsoidal bound
  - finding min. bounding ellipsoid
  - semidefinite programming relaxation
  - conservative approximation