Robust Truss Topology Optimization with Discrete Design Variables via Mixed Integer Programming

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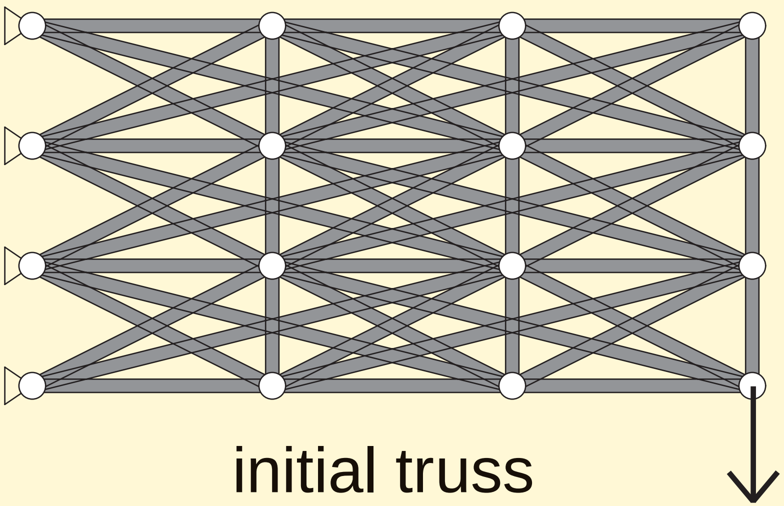
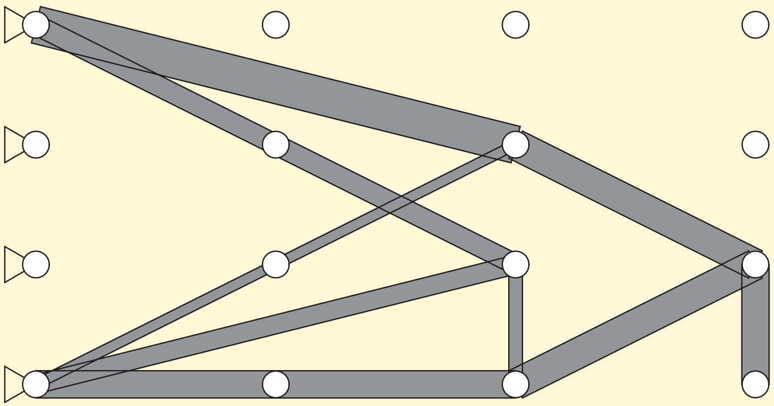
Motivation: Robust Topology Optimization

- truss topology optimization
  - under single load $\longrightarrow$ opt. solution is often unstable
  - $\longrightarrow$ robust optimization is required



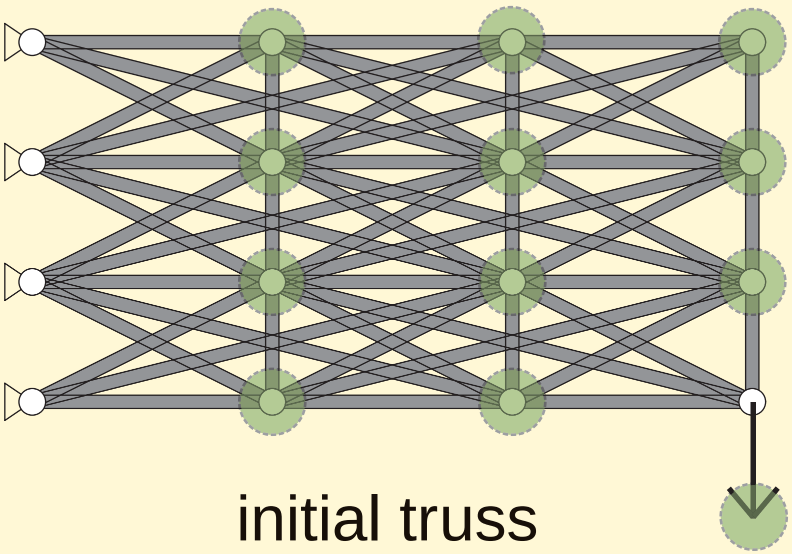
initial truss



conventional opt. solution

Motivation: Robust Topology Optimization

- truss topology optimization [Ben-Tal & Nemirovski 97]
  - uncertain loads at all nodes, in all directions
  - minimize the compliance in the worst-case



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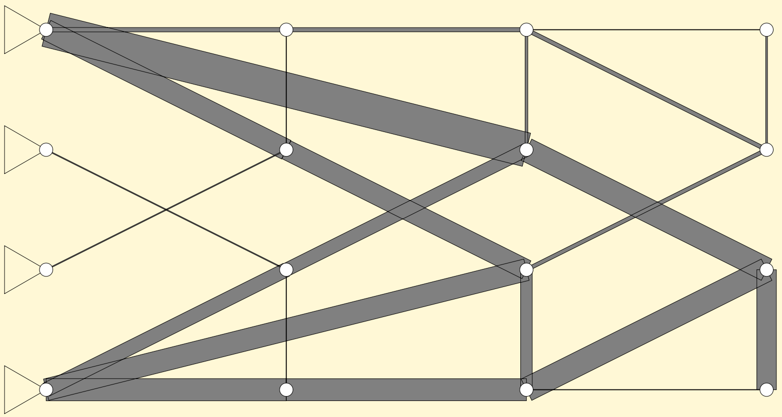

robust opt. solution

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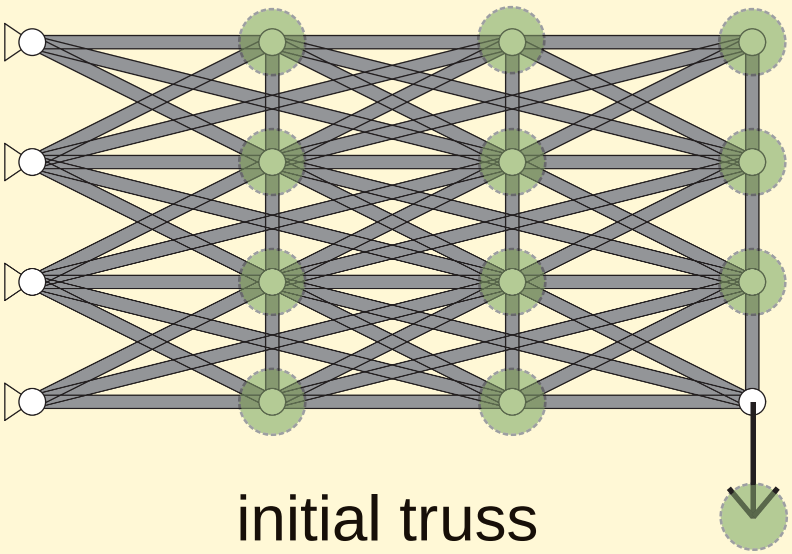

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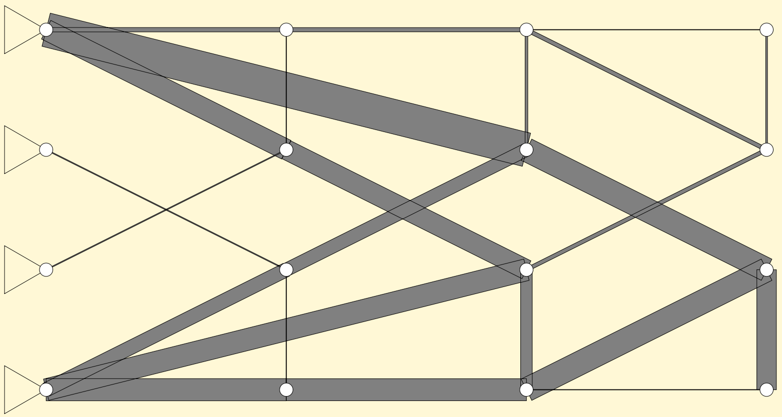

robust opt. solution

- stable solution is always obtained
- all nodes remain $\Rightarrow$ topology is not (necessarily) optimal

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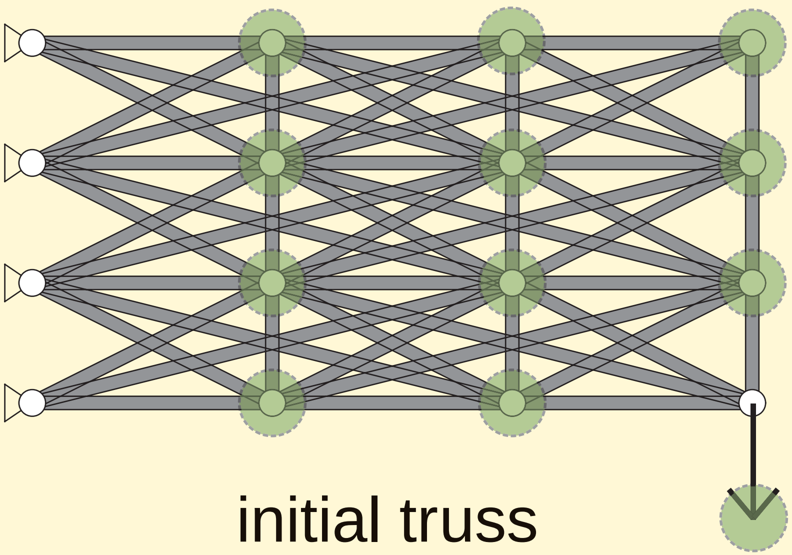

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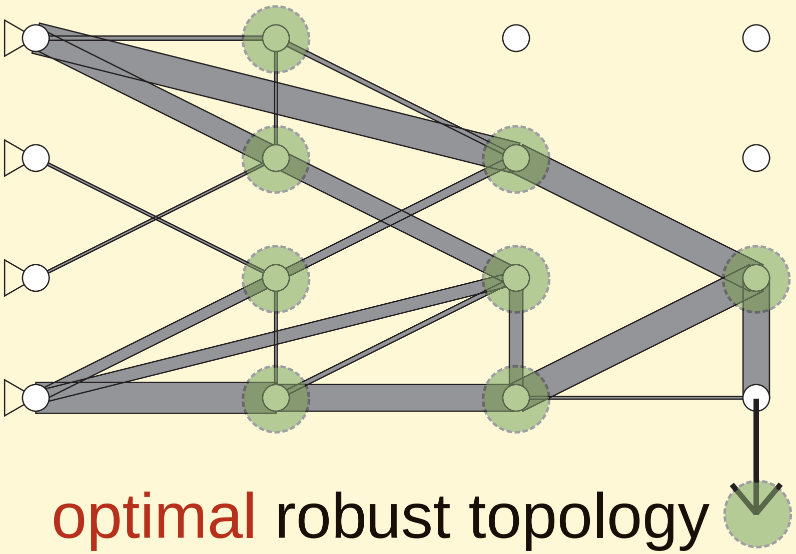

optimal robust topology

- → propose a robust truss optimization formulation considering the variation of topology

Existing Methods of Robust Truss Optimization

- probabilistic approach
- possibilistic approach
  - convex model [Ben-Haim & Elishakoff 90]
    - [Elishakoff, Haftka & Fang 94] [Ganzerli & Pantelides 98], [Au, Cheng, Tham & Zheng 03] [Jiang, Han & Liu 07]
  - 1st-order approximation
    - [Lee & Park 01]

topology does not change ($x_i \geq \epsilon$)

  - semidefinite program
    - compliance [Ben-Tal & Nemirovski 97] all nodes remain
    - stress [Kanno & Takewaki 06]

stress constraints are not removed

Robust Optimization (Possibilistic Approach)

- nominal (conventional) truss optimization

$$\begin{array}{ll} \min_{\boldsymbol{x} \in \mathcal{X}} & vol(\boldsymbol{x}) \\ \text{s.t.} & g_q(\boldsymbol{u}) \le 0, \quad \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f} \end{array}$$

  - $g_q(\boldsymbol{u}) \le 0$: constraint on the mechanical performance
  - $\boldsymbol{x}$: cross-sectional areas, $\boldsymbol{K}(\boldsymbol{x})$: stiffness matrix,
    $\boldsymbol{u}$: displacement, $\boldsymbol{f}$: external load

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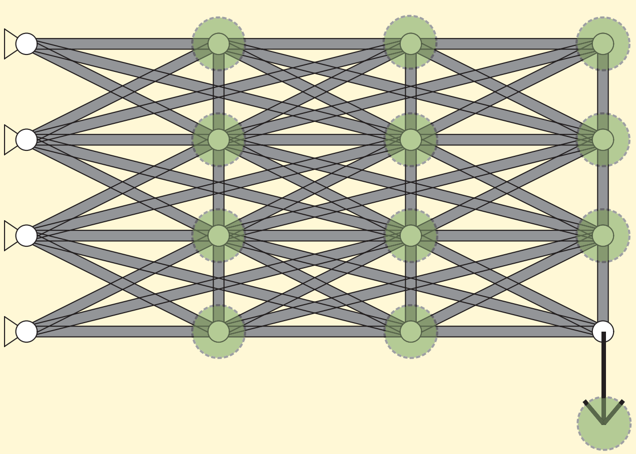
- robust constraint

$$\max_{\boldsymbol{u}} \{ g_q(\boldsymbol{u}) \mid \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} \in \bar{\mathcal{F}} \} \leq 0 \qquad (\spadesuit)$$

- $\bar{\mathcal{F}}$: uncertainty set of external loads



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Uncertainty Model

- nominal load : $\tilde{\boldsymbol{f}} \in \mathbf{R}^n$
- uncertainty set

$$\bar{\mathcal{F}} = \{\tilde{\boldsymbol{f}} + \boldsymbol{F}_0 \boldsymbol{\zeta} \mid \alpha \geq \|\boldsymbol{\zeta}_j\| \ (\forall j)\}$$

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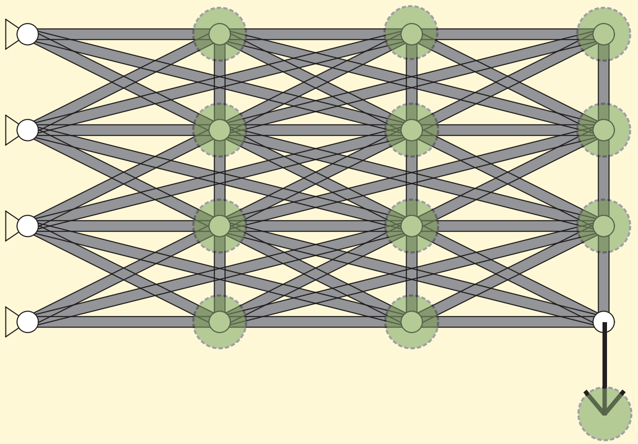
  - $\boldsymbol{F}_0$: constant matrix
  - $\alpha \geq 0$: level of uncertainty
  - j : index of node
  - uncertainty at all nodes



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$$\mathcal{F}(\boldsymbol{s}) = \{\tilde{\boldsymbol{f}} + \boldsymbol{F}_0 \boldsymbol{\zeta} \mid \alpha s_j \geq \|\boldsymbol{\zeta}_j\| \ (\forall j)\}$$

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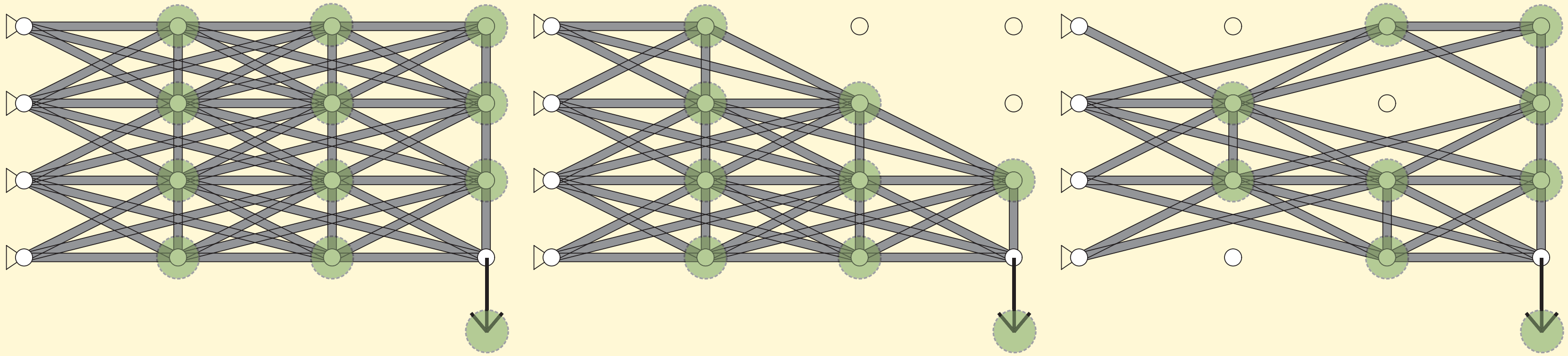
  - $s_j = \begin{cases} 1 & \text{if the } j\text{th node exists} \\ 0 & \text{if the } j\text{th node is removed} \end{cases}$



Worst-Case Detection

- worst case of response g_q

$$\max_{\boldsymbol{u}}\{g_q(\boldsymbol{u}) \mid \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}(\boldsymbol{s})\} \qquad (\spadesuit)$$

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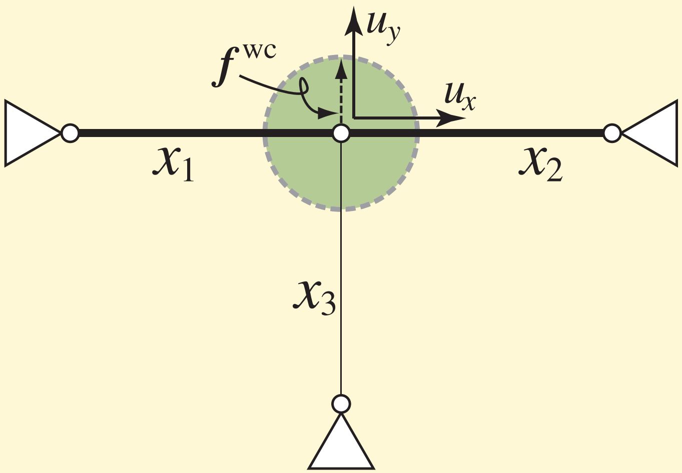
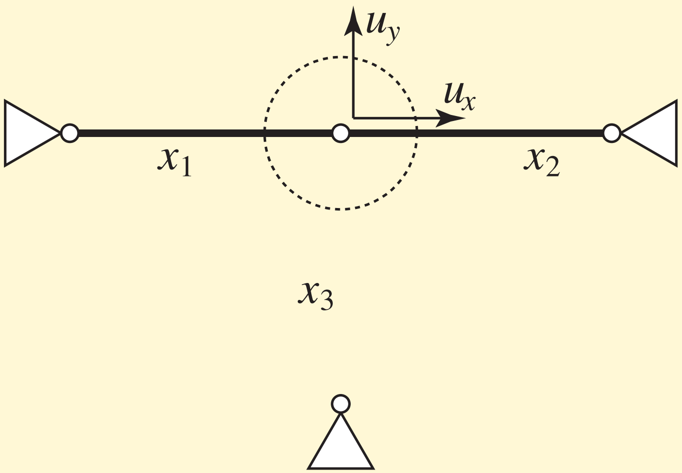
- counter-example : $\max_{\boldsymbol{u}} u_y$?





$x_3 = \varepsilon$: ($\spadesuit$) provides w.c. $\quad$ $x_3 = 0$: w.c. is infeasible for ($\spadesuit$)

Worst-Case Detection

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-

a truss is stable	$\Rightarrow$	($\spadesuit$) provides w.c.
otherwise		not necessarily

- stability constraint is necessary
  for robust truss topology optimization

Discrete Design Variables

- member cross-sectional area x_i

$$x_i \in \{0, \xi_1, \dots, \xi_k\}$$

- 0–1 variables t_{ip} e.g., [Stolpe & Svanberg 03]

$$x_i = \sum_{p=1}^{k} \xi_p t_{ip}, \quad \sum_{p=1}^{k} t_{ip} \leq 1$$

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- indices of area t_{ip} & index of node

$$t_{ip} \leq s_j \leq 1$$

$$s_j \leq \sum_{i \in \mathcal{I}_j} \sum_{p=1}^{k} t_{ip}$$

  - $i \in \mathcal{I}_j \iff$ member i is connected to node j

Stress Constraints

- worst case of stress σ_i

$$\max_{\boldsymbol{u}}\{\sigma_i(\boldsymbol{u}) \mid \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}(\boldsymbol{s})\} \leq \bar{\sigma} \qquad (\diamondsuit \text{a})$$

$$\min_{\boldsymbol{u}}\{\sigma_i(\boldsymbol{u}) \mid \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}(\boldsymbol{s})\} \geq -\bar{\sigma} \qquad (\diamondsuit \text{b})$$

- constraints ($\diamondsuit$) are rewritten by using the KKT conditions (with several Lagrange multipliers)

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- constraints ($\diamondsuit$) are rewritten by using the KKT conditions (with several Lagrange multipliers)

- $x_i = 0 \ \Rightarrow$ stress constraint should be removed

$$|\sigma_i(\boldsymbol{u})| \leq \bar{\sigma} + M(1 - t_i), \qquad t_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{if } x_i = 0 \end{cases}$$

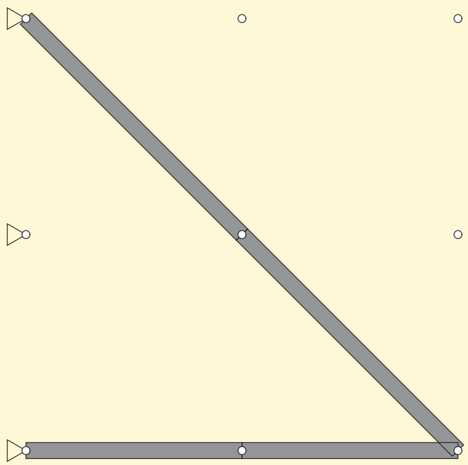
e.g., [Stolpe & Svanberg 03]

- constraints on the associated Lagrange multipliers should also be removed

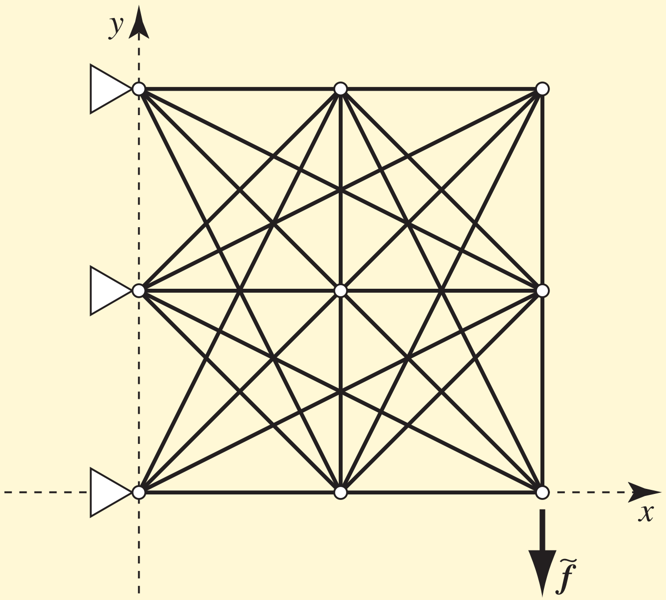
Global Optimization

- MIP (mixed integer programming) formulation
  - discrete cross-sectional areas
  - topology-dependent uncertainty model
    - uncertainty loads exist only at existing nodes
  - stress constraint in the worst case
    - imposed only to the existing members
  - stability constraint
    - a necessary condition is considered

Ex.) 26-member truss ($\mathcal{X} = \{0, 20\}^m$, $\|\tilde{f}\| = 10.0$)



nominal opt.



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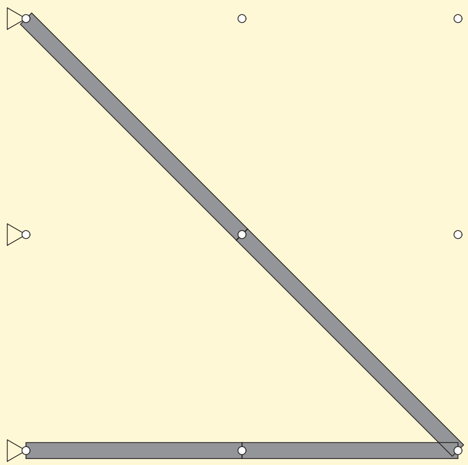

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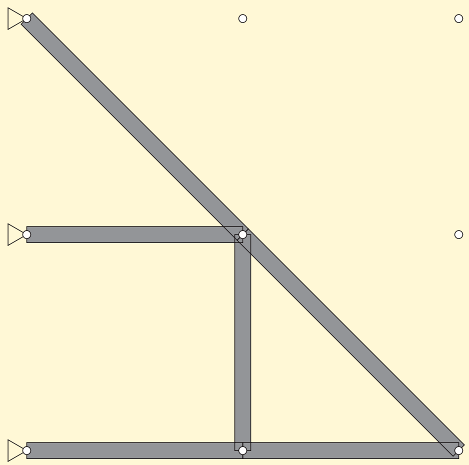

robust opt. $(\alpha = 1.0)$

- robust optimal topology depends on the level of uncertainty

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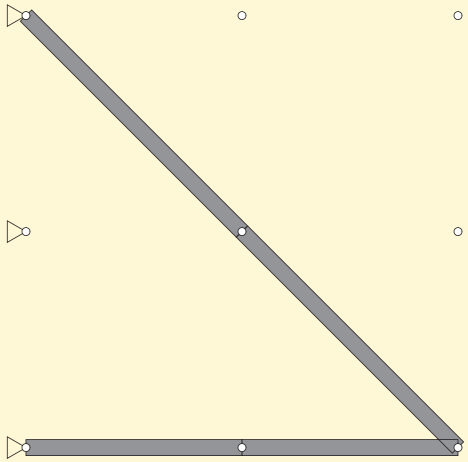

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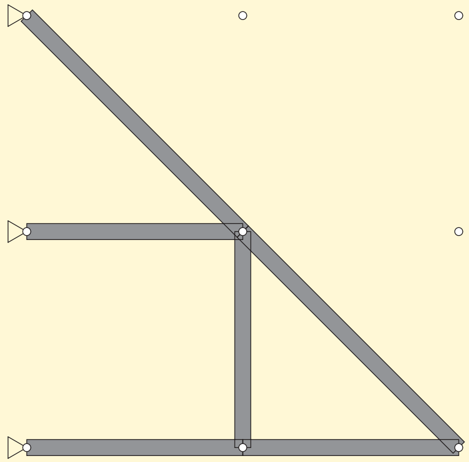

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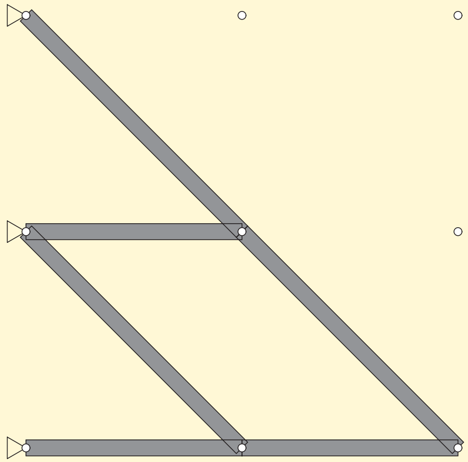
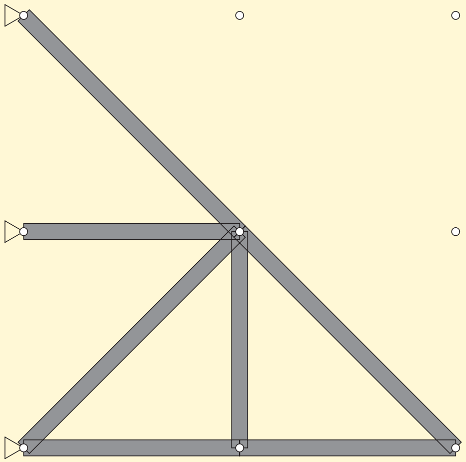


robust opt. $(\alpha = 1.5)$



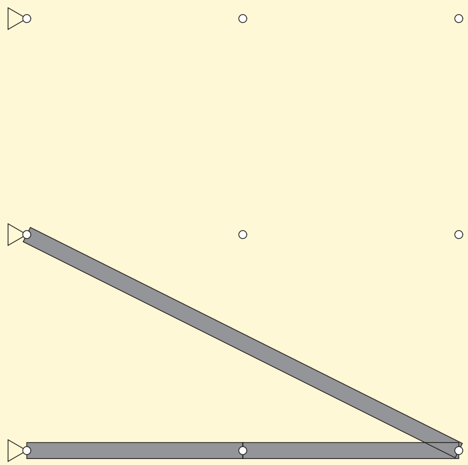
robust opt. $(\alpha = 3.0)$

Ex.) 26-member truss (computational result)

- MIP solver: CPLEX Ver. 11.2

α	Vol. (cm^3)	CPU (s)	# of Nodes
nominal	9656.9	≤ 0.1	24
1.0	13656.9	4029.8	13348
1.5	14485.3	9241.6	92693
3.0	16485.3	630338.8	4231298
3.3	16485.3	29507.6	133683

Ex.) 26-member truss ($\mathcal{X} = \{0, 20\}^m$, $\|\tilde{f}\| = 7.5$)



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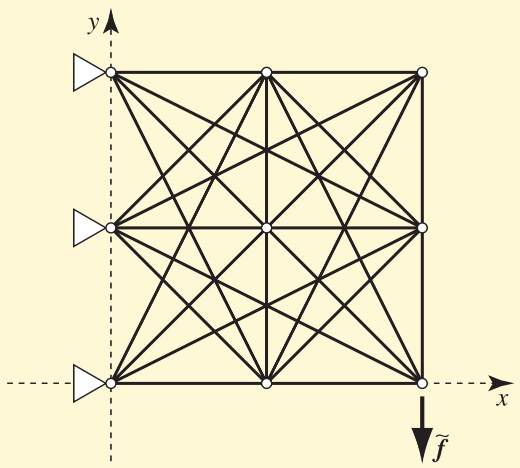

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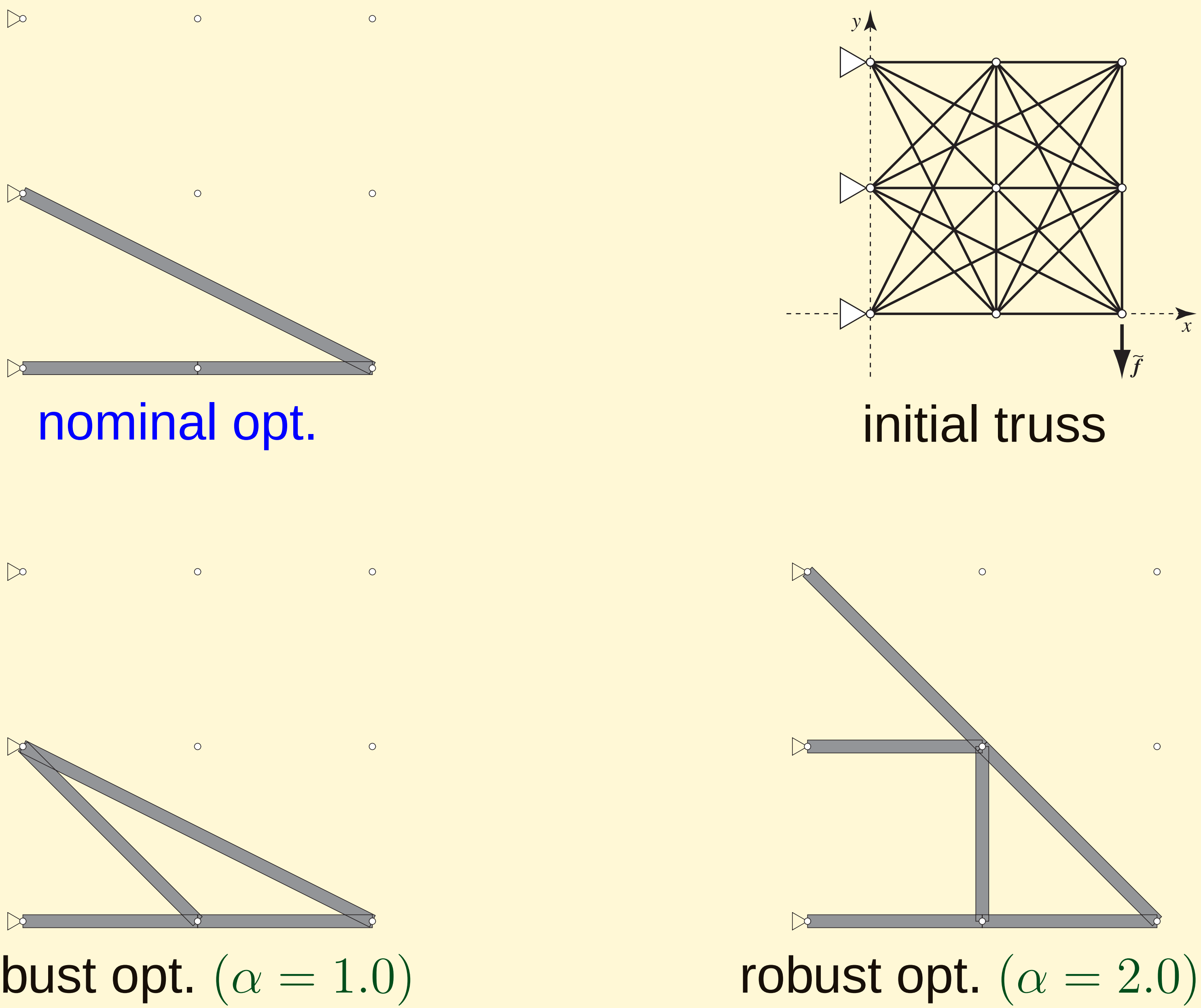
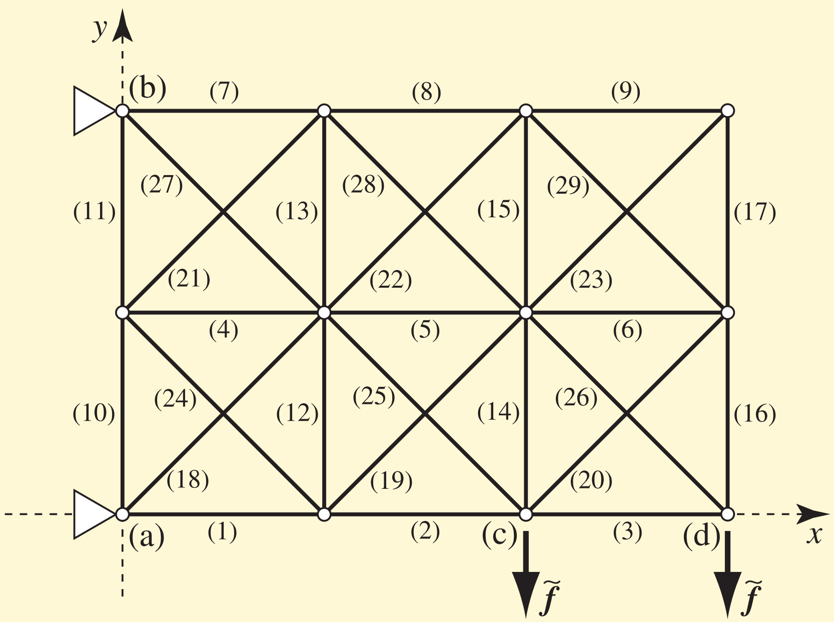


Ex.) 29-member truss ($\alpha = 1.0$)



- robust optimal topology depends on
  the set of candidates of cross-sectional areas

- robust optimal topology is not necessarily unique

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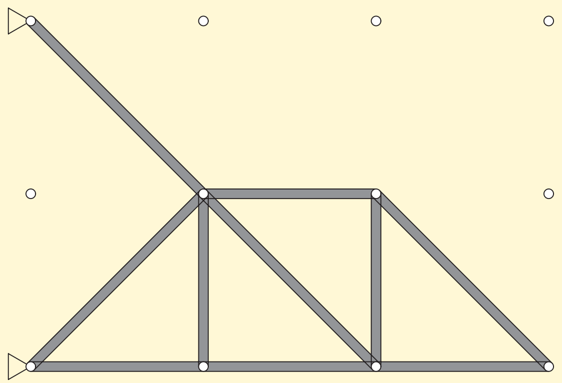
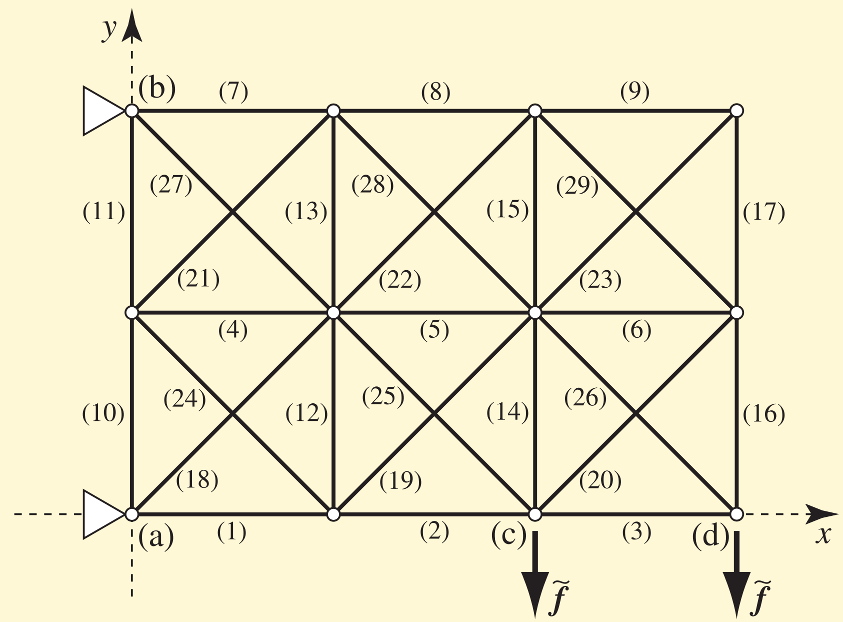


robust opt.

$(\mathcal{X} = \{0, 10\}^m)$



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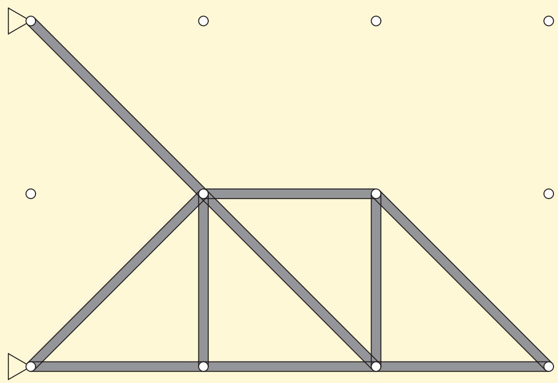

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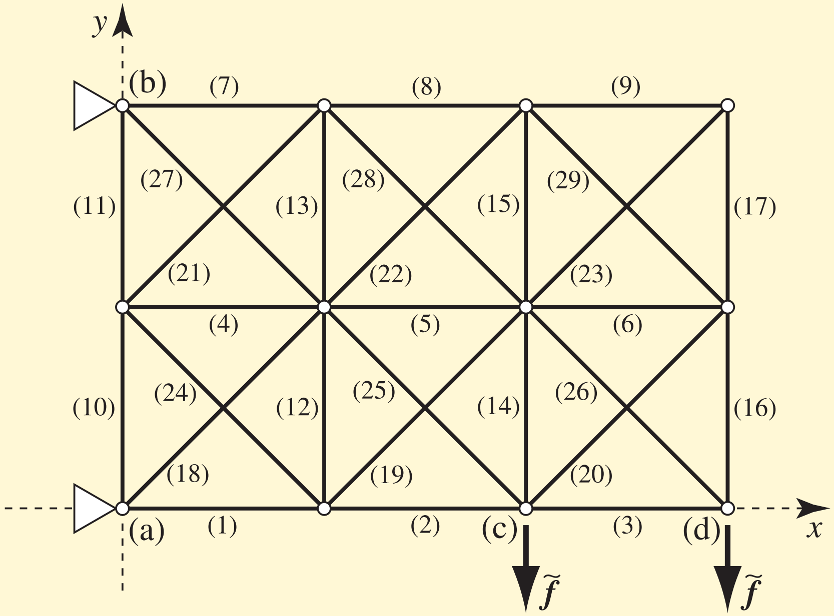

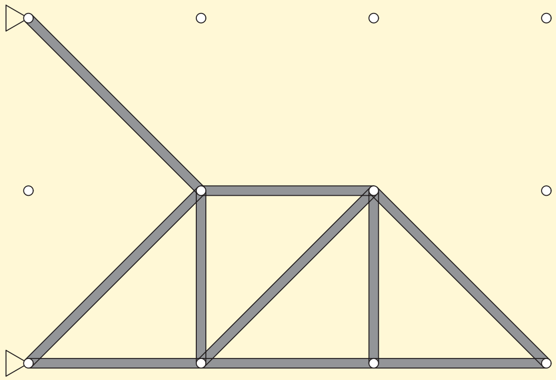

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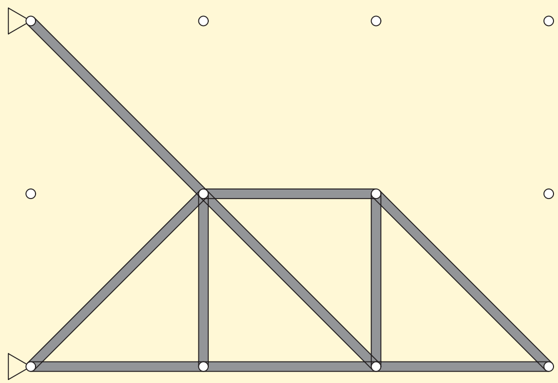

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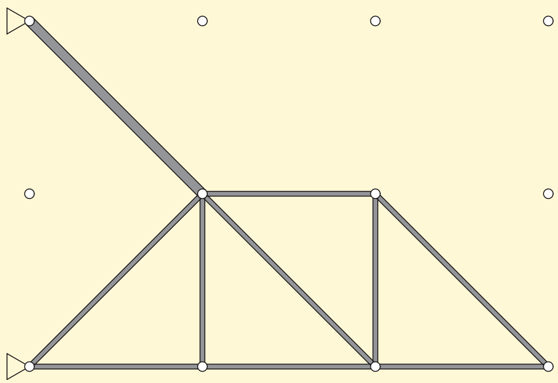

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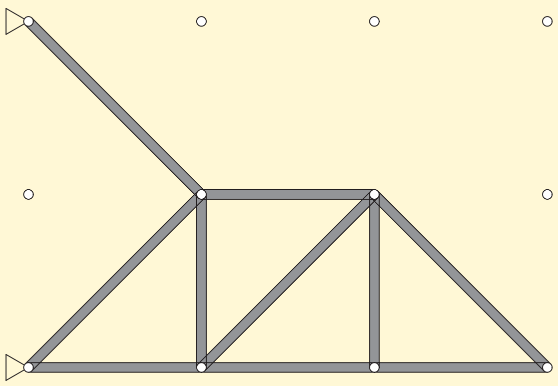

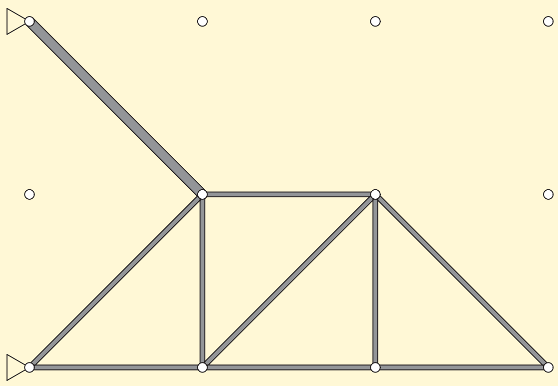

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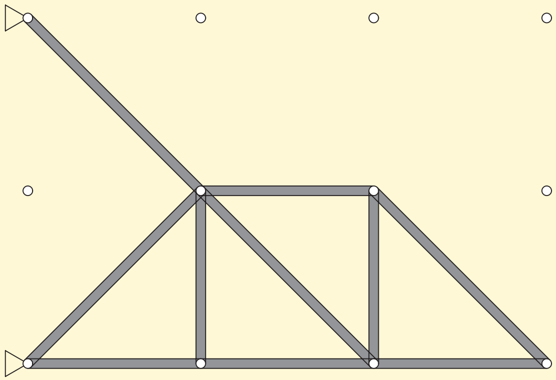

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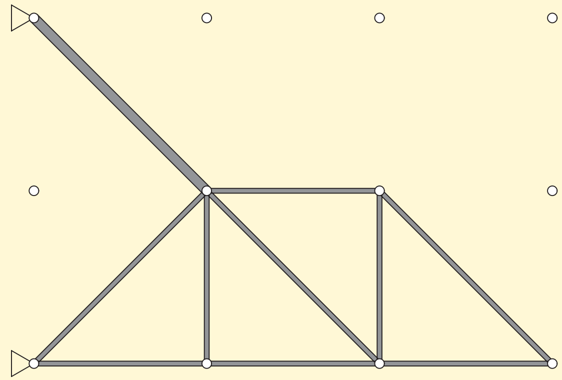

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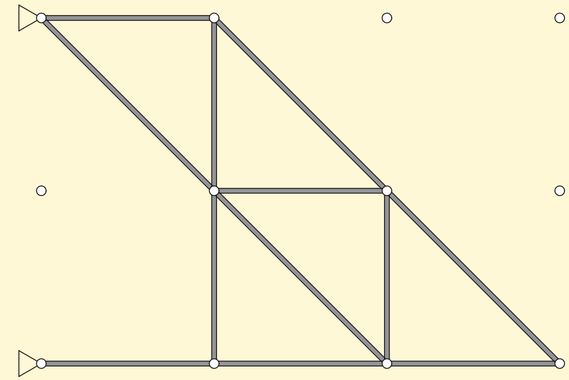
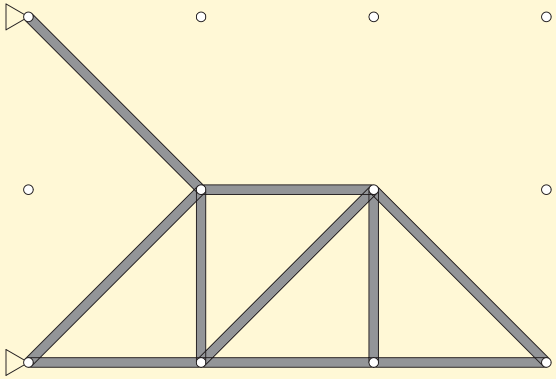
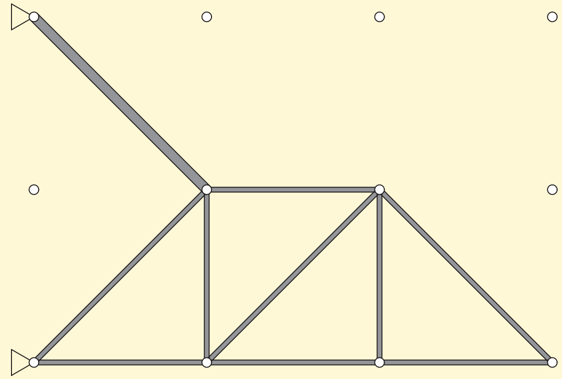
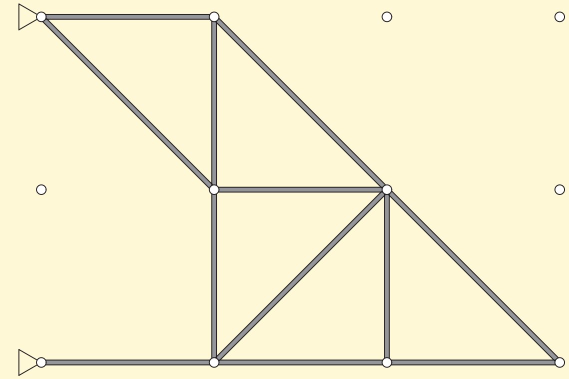


robust opt.
($\mathcal{X} = \{0, 5, 15\}^m$)







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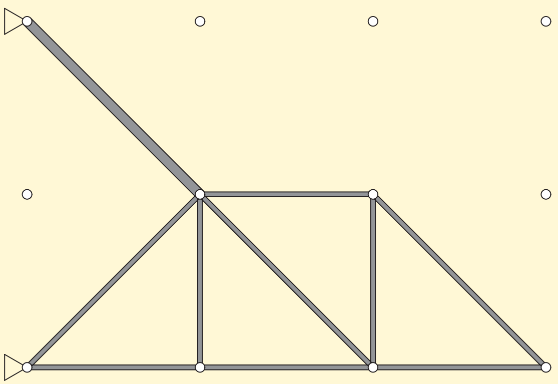
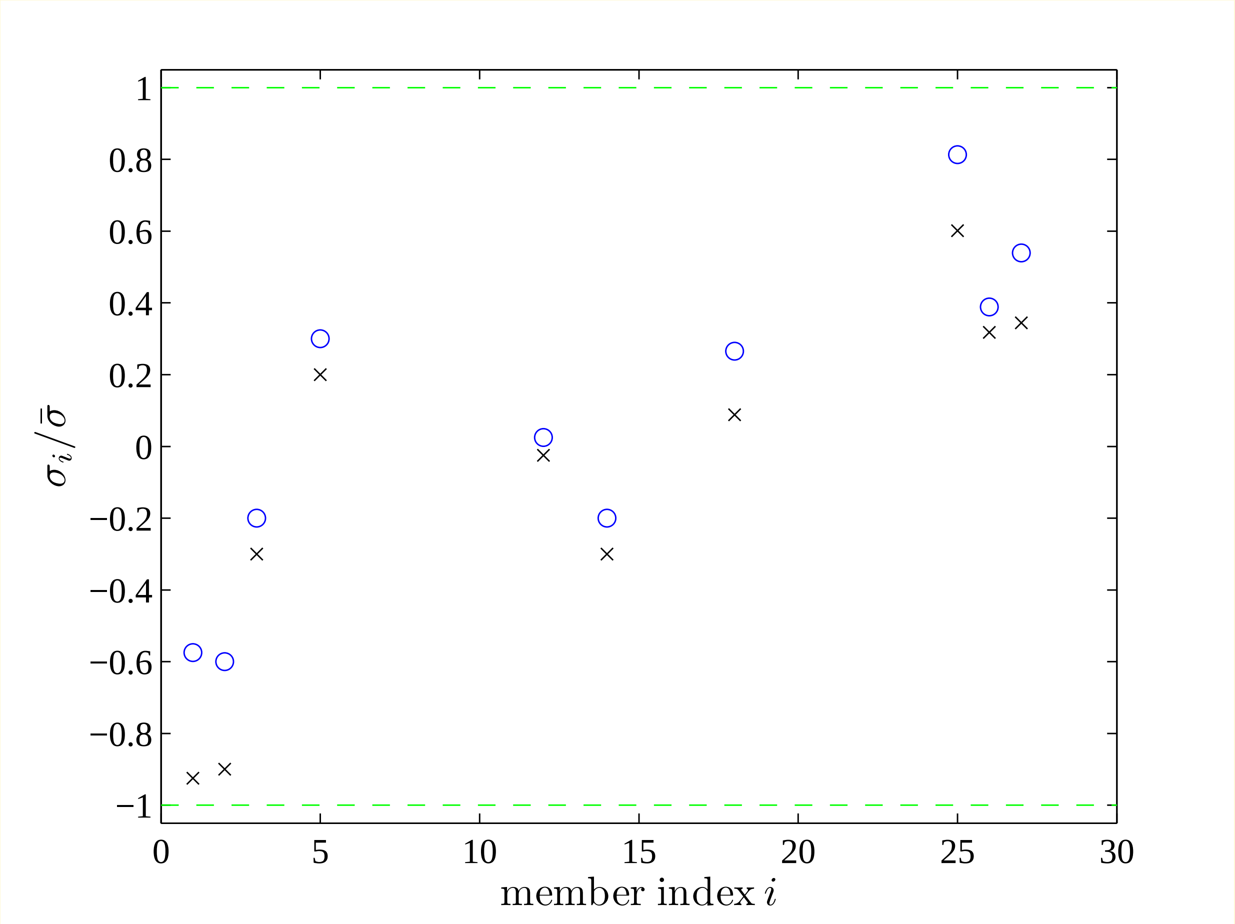


robust opt.

$(\mathcal{X} = \{0, 5, 10\}^m)$



worst-case member stresses

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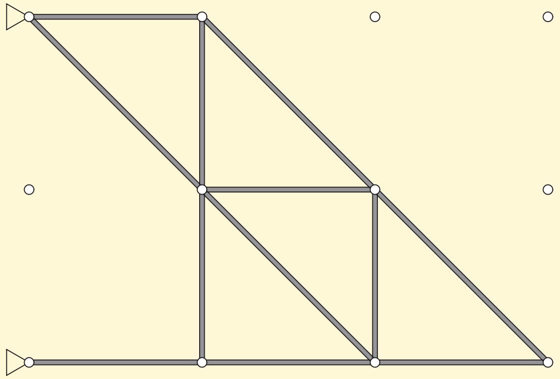
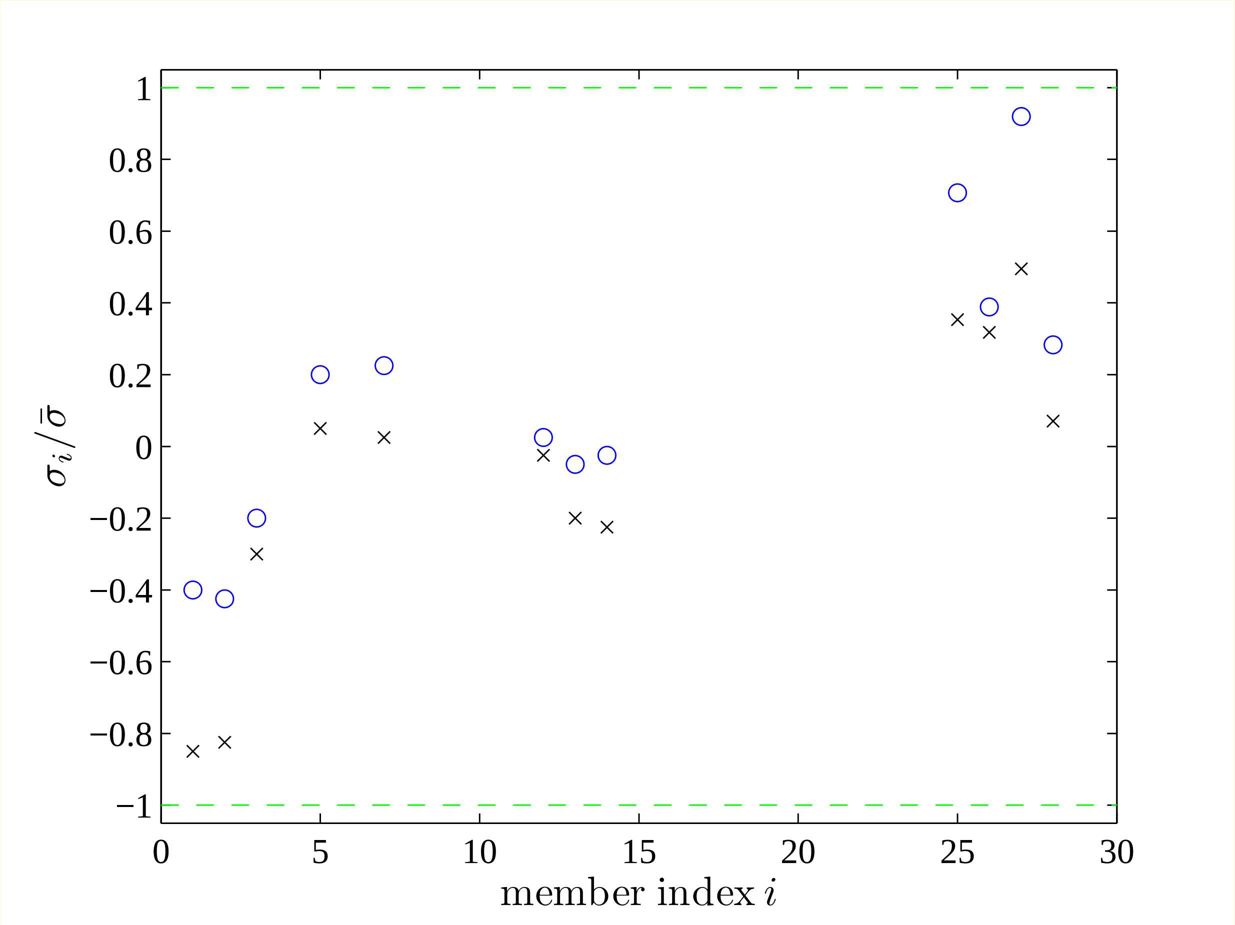


robust opt.

$(\mathcal{X} = \{0, 5, 15\}^m)$



worst-case member stresses

Conclusions

- robust truss optimization
  - topology optimization
  - topology-dependent uncertainty model
    uncertain loads at all existing nodes
- stress constraints
  - $-\bar{\sigma} \leq$ (stress in the worst case) $\leq \bar{\sigma}$
    for all existing members
  - constraint on stability is required
- global optimization
  - Mixed Integer Programming formulation