Mixed Integer Programming for Finding Tensegrity Structures

Shintaro Ehara   Yoshihiro Kanno *

University of Tokyo (Japan)

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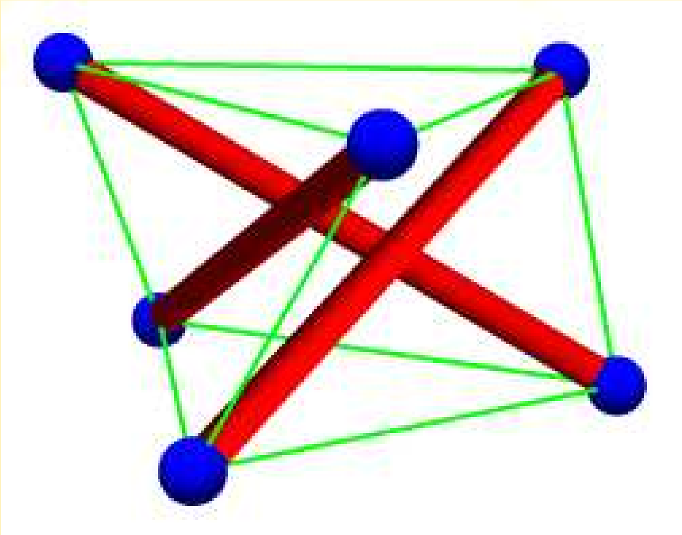
tensegrity — definition

- [Buckminster Fuller 75]
- pin-jointed structure
  - cable — tensile force
  - strut — compressive force
- self-equilibrium condition
  - with prestresses
- discontinuity of struts
  - each node has at most one strut
- topology (→)



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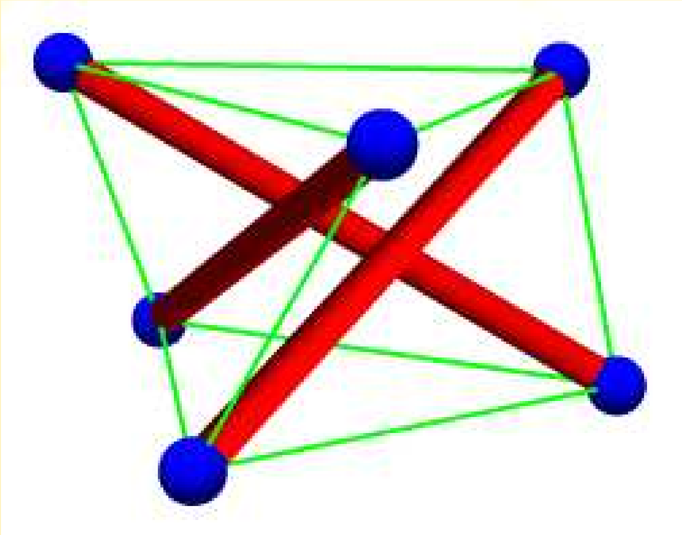

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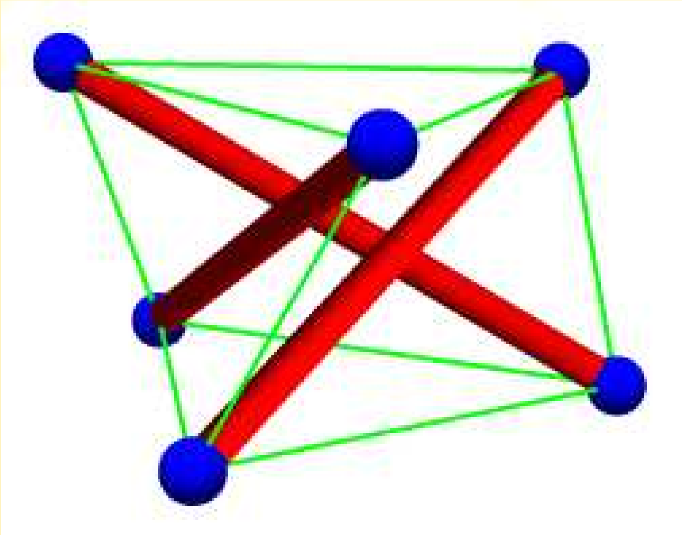

form-finding / topology

- topology
  - connectivity
  - labeling — "cable" or "strut"
- given a topology — find locations of nodes
  - group-theoretic symmetry
    [Connelly & Terrell 95] [Connelly & Back 98]
  - rotational symmetry
    [Sultan, Corless & Skelton 02] [Masic, Skelton & Gill 05]
  - mathematical programming
    [Zhang, Ohsaki & Kanno 06]
- given locations of nodes — find a topology

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    [Zhang, Ohsaki & Kanno 06]
- given locations of nodes — find a topology $\leftarrow$ (!)

mixed integer program (MIP)

- LP (linear program)

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \mathrm{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

variables : $\boldsymbol{x}$ (continuous)

mixed integer program (MIP)

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variables : $\boldsymbol{x}$ (continuous)

- MIP (mixed integer program)

$$\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{t} \\ \text{s.t.} & \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{t} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}, \\ & t_i = 0 \text{ or } 1 \end{array}$$

variables : $\boldsymbol{x}$ (continuous)

$\boldsymbol{t} \in \{0,1\}^m$ (integer)

MIP and algorithms

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{t}} \quad & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{t} \\ \text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{t} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}, \\ & t_i \in \{0,1\} \end{aligned}$$

- $t_i \in \{0,1\} \quad \Leftrightarrow \quad t_i = 0$ or 1
  - integer (or binary, discrete) variable
  - LP relaxation: $0 \leq t_i \leq 1$
- global optimization
- application

MIP and algorithms

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{t}} \quad & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{t} \\ \text{s.t.} \quad & \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{t} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}, \\ & t_i \in \{0, 1\} \end{aligned}$$

- $t_i \in \{0, 1\} \quad \Leftrightarrow \quad t_i = 0$ or 1
- global optimization
  - branch-and-bound method, cutting plane method
  - branch-and-cut method
  - software packages [CPLEX], [SCIP], [GLPK], etc
- application

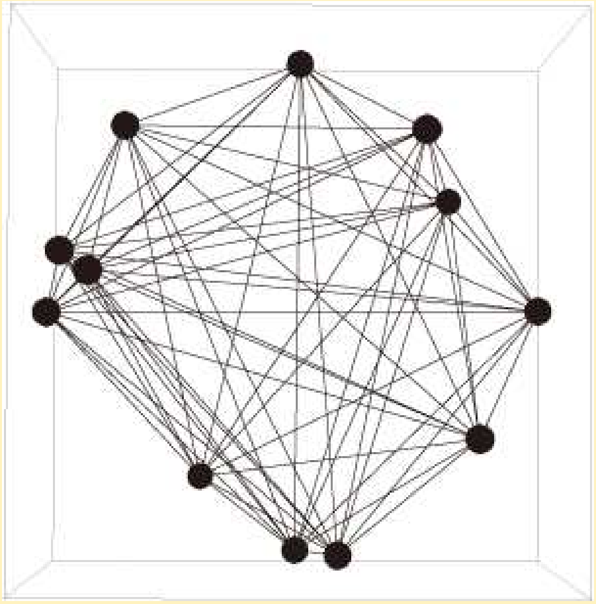
MIP and algorithms

$$\begin{array}{ll} \min_{\boldsymbol{x},\boldsymbol{t}} & \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{c}_2^{\mathrm{T}}\boldsymbol{t} \\ \mathrm{s.t.} & \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{A}_2\boldsymbol{t} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}, \\ & t_i \in \{0,1\} \end{array}$$

- $t_i \in \{0,1\} \quad \Leftrightarrow \quad t_i = 0$ or 1
- global optimization
- application
  - combinatorial optimization
  - discrete optimization of structures
    [Stolpe & Svanberg 03], [Rasmussen & Stolpe 08]
  - worst-case analysis of uncertain structures
    [Kanno & Takewaki 07], [Guo, Bai & Zhang 08]

two-stage algorithm

- ground structure method / structural optimization



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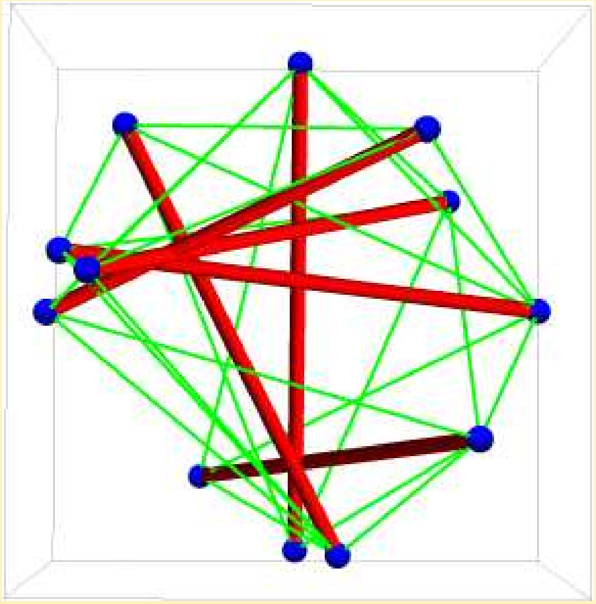
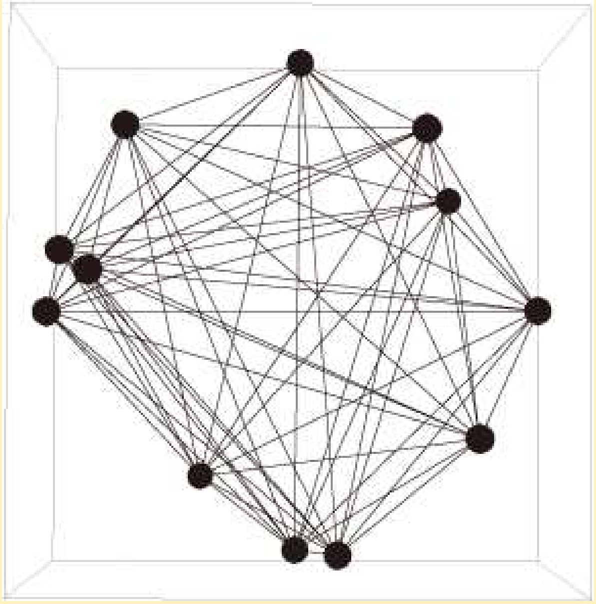


- given:
  - candidate members
  - locations of nodes
- variables: member cross-sectional areas

two-stage algorithm

- ground structure method / structural optimization



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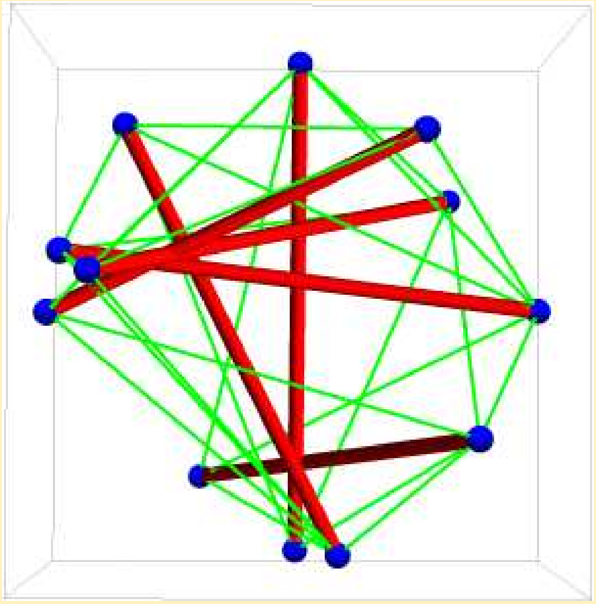


- two stages:
  - MIP-1: find a tensegrity (possibly with many cables)
  - MIP-2: remove redundant cables
- requires no information of topology in advance

discontinuity condition of struts

$$\sum_{i \in E(n_j)} t_i \leq 1, \qquad \forall \text{nodes} \qquad (\clubsuit 1)$$

$$-Mt_i \leq q_i \leq M(1 - t_i) - \varepsilon, \qquad \forall \text{members} \qquad (\clubsuit 2)$$

- $q_i \begin{cases} > 0 & : \text{cable} \\ = 0 & : \text{removed} \\ < 0 & : \text{strut} \end{cases} \qquad t_i \in \{0, 1\}$
- $M \gg \varepsilon > 0$: constants

discontinuity condition of struts

$$\sum_{i \in E(n_j)} t_i \le 1, \qquad \forall \text{nodes} \qquad (♣1)$$

$$-Mt_i \le q_i \le M(1 - t_i) - \varepsilon, \qquad \forall \text{members} \qquad (♣2)$$

- $q_i \begin{cases} > 0 & : \text{ cable} \\ = 0 & : \text{ removed} \\ < 0 & : \text{ strut} \end{cases} \qquad t_i \in \{0, 1\}$
- $M \gg \varepsilon > 0$: constants
- ♣2: $t_i = 1 \iff q_i < 0$ (i.e. strut)
- ♣1: each node can have at most one strut
- max $\sum t_i \iff$ max "# of struts"

MIP-1: max "# of struts"

$$\max_{\boldsymbol{q},\boldsymbol{t}} \quad \sum_{i \in E} t_i$$

$$\text{s.t.} \quad \boldsymbol{H}\boldsymbol{q} = \boldsymbol{0}, \qquad (\diamondsuit)$$

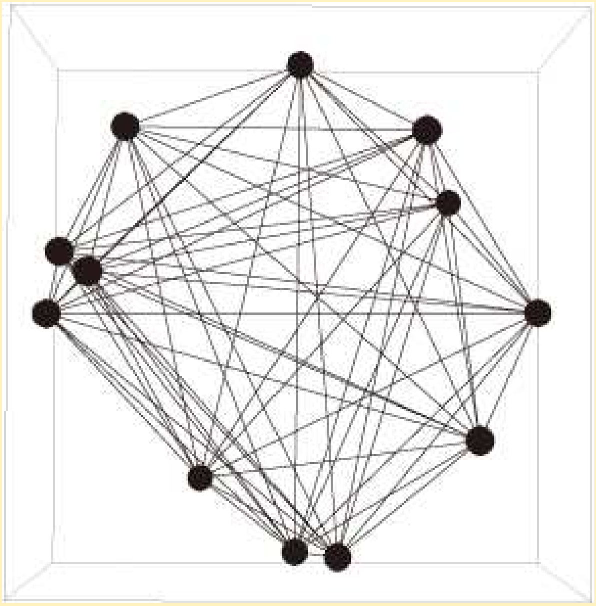
$$\sum_{i \in E(n_j)} t_i \leq 1, \qquad \forall j \in V \qquad (\clubsuit 1)$$

$$-Mt_i \leq q_i \leq M(1 - t_i) - \varepsilon, \qquad \forall i \in E \qquad (\clubsuit 2)$$

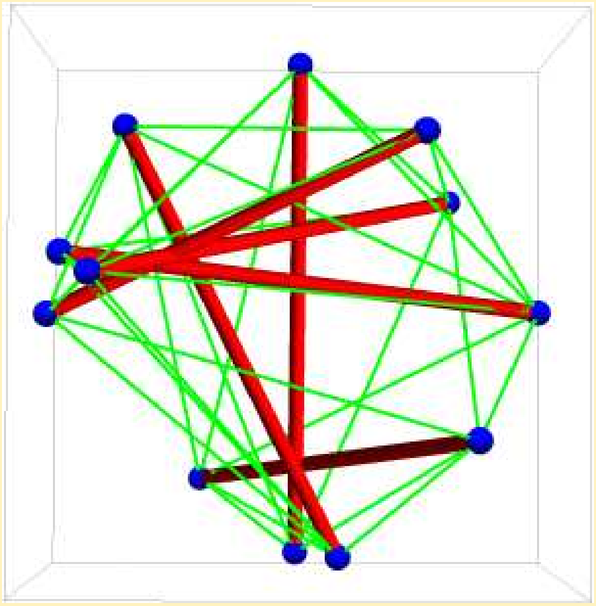
$$t_i \in \{0, 1\}, \qquad \forall i \in E$$

variables :	q_i	(axial force)
	t_i	(label)
given:	E	(member candidates)
	V	(locations of nodes)
constraints:	$\diamondsuit$	(self-equilibrium)
	$\clubsuit$	(discontinuity of struts)

solution of MIP-1



$\longrightarrow$



- satisfies the discontinuous condition of struts
- possibly includes many cables
  $\rightarrow$ MIP-2: reduction of number of cables

MIP-2: min "# of cables"

$$
\begin{aligned}
\min_{\boldsymbol{q},\boldsymbol{y}} \quad & \sum_{i \in E} y_i \\
\text{s.t.} \quad & \boldsymbol{H}\boldsymbol{q} = \boldsymbol{0}, && && (\diamondsuit) \\
& q_i \leq -\varepsilon, && \forall i \in E_{\mathrm{strut}} && (\heartsuit) \\
& 0 \leq q_i \leq M y_i, && \forall i \in E_{\mathrm{cable}} && (\spadesuit) \\
& y_i \in \{0, 1\}, && \forall i \in E
\end{aligned}
$$

variables :	q_i	(axial force)
	y_i	(label)
given:	E_{strut}	(struts)
	E_{cable}	(candidates of cables)
constraints:	$\diamondsuit$	(self-equilibrium)

label of cable

$$q_i \leq -\varepsilon, \qquad \forall\text{struts} \qquad (\heartsuit)$$
$$0 \leq q_i \leq M y_i, \qquad \forall\text{cables} \qquad (\spadesuit)$$

- $q_i \begin{cases} > 0 & : \text{cable} \\ = 0 & : \text{removed} \\ < 0 & : \text{strut — fixed } (\heartsuit) \end{cases}$ $\qquad y_i \in \{0, 1\}$
- $M \gg \varepsilon > 0$: constants

label of cable

$$q_i \le -\varepsilon, \qquad \forall\text{struts} \qquad (\heartsuit)$$
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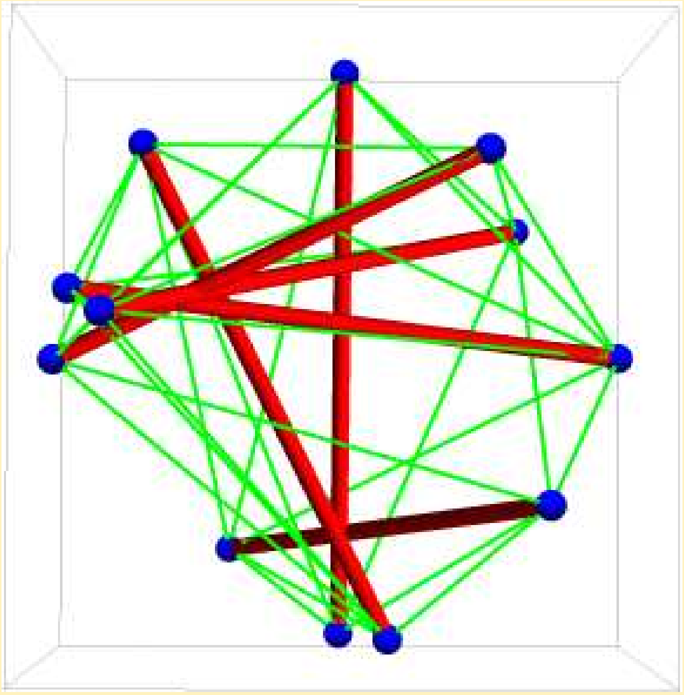
- $q_i \begin{cases} > 0 & : \text{cable} \\ = 0 & : \text{removed} \\ < 0 & : \text{strut — fixed } (\heartsuit) \end{cases}$ $\qquad y_i \in \{0, 1\}$
- $M \gg \varepsilon > 0$: constants
- min $\sum y_i$
  - ♠: $y_i = 1 \;\Leftrightarrow\; q_i > 0$ (i.e. cable)
  - $\sum y_i \;=\;$ "# of struts"

solution of MIP-2



30 cables

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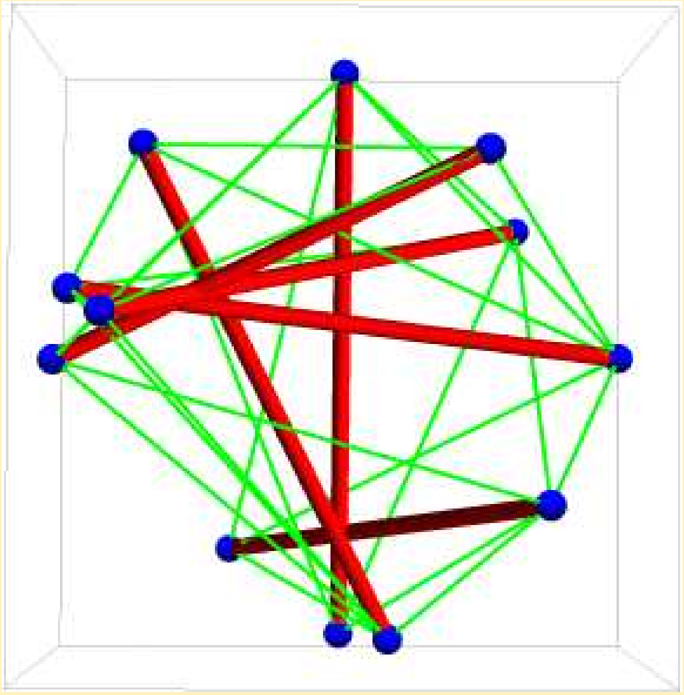


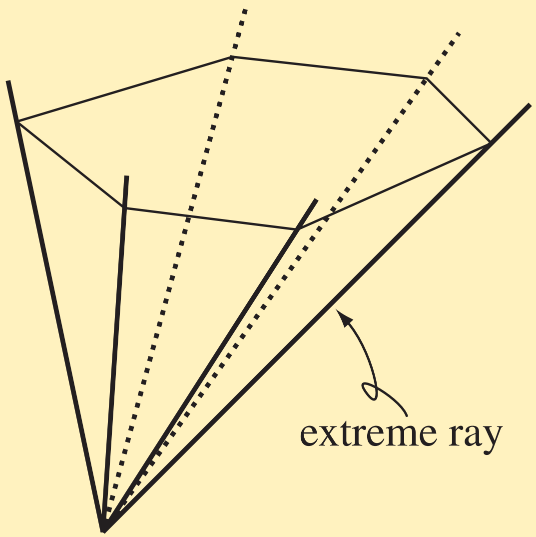
25 cables

- 6 struts
- satisfies the discontinuous condition of struts
- includes no “redundant” cables

enumeration of tensegrities

- set of tensegrities (self-equilibrium modes)

$$\mathcal{T} = \left\{ \boldsymbol{q} \;\middle|\; \begin{array}{l} \boldsymbol{Hq} = \boldsymbol{0} \\ q_i \geq 0 \ (\forall i \in E_{\mathrm{cable}}) \\ q_j \leq 0 \ (\forall j \in E_{\mathrm{strut}}) \end{array} \right\}$$



- $\boldsymbol{q} \in \mathbb{R}^{|E|}$
  $(E := E_{\mathrm{cable}} \cup E_{\mathrm{strut}})$
- $\mathcal{T} \;\leftrightarrow\;$ polyhedral cone in $\mathbb{R}^{|E|}$
- (minimal) tensegrity $\leftrightarrow$ extreme ray

enumeration of tensegrities

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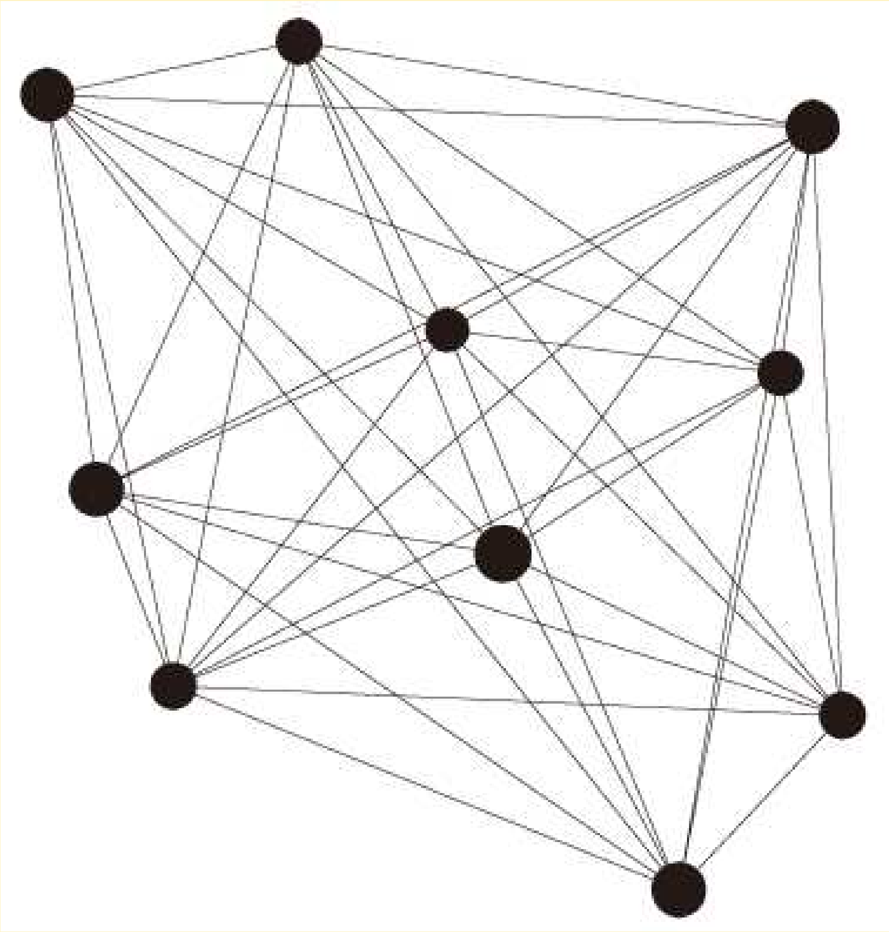
- enumeration of extreme rays:
  - double description method
    [Motzkin, Raiffa, Thompson, & Thrall 53] [Fukuda & Prodon 96]
  - reverse search
    [Avis & Fukuda 96]

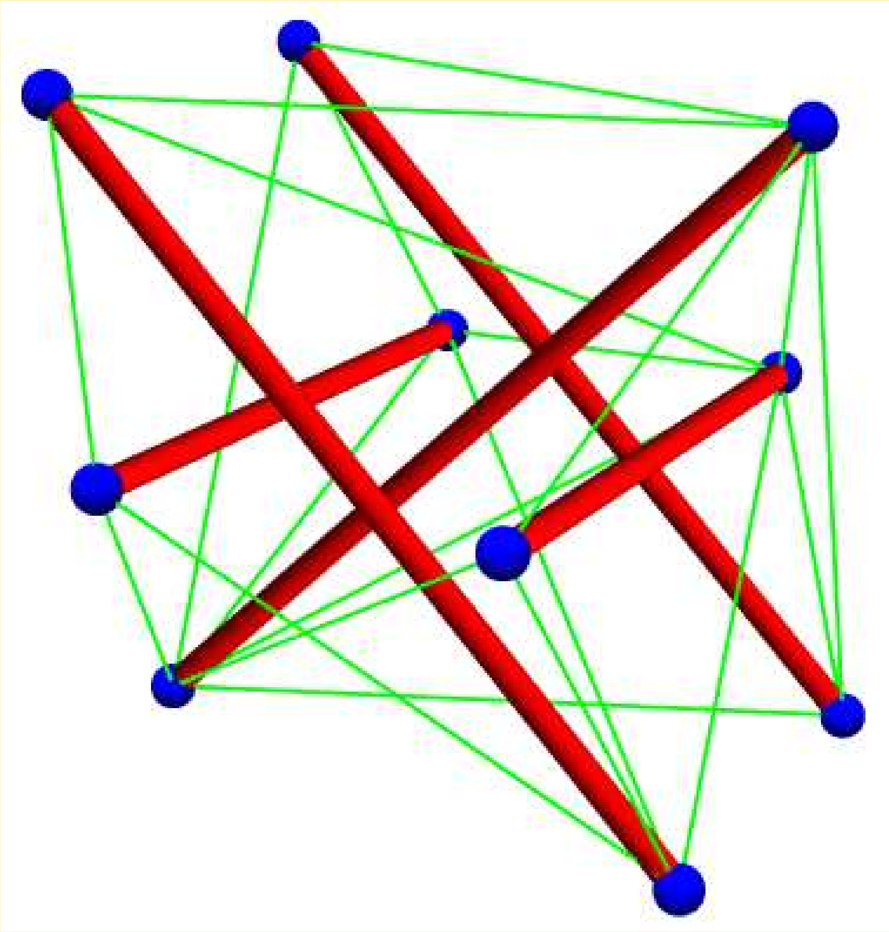
ex.) symmetric configuration

- 10 nodes
- 41 members
  (perfect graph)
- CPLEX (ver. 11.2)



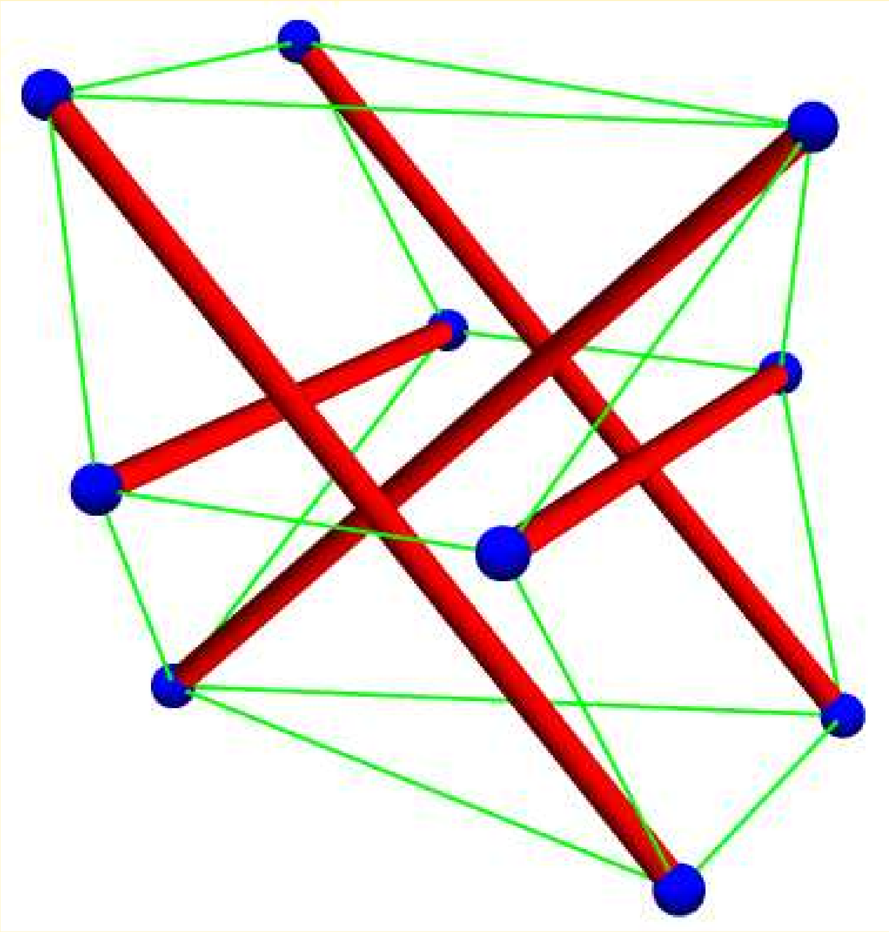
ex.) symmetric configuration

- 10 nodes
- 41 members
  (perfect graph)
- CPLEX (ver. 11.2)
- 5 struts
- solution of MIP-1
  $\rightarrow$ 18 cables

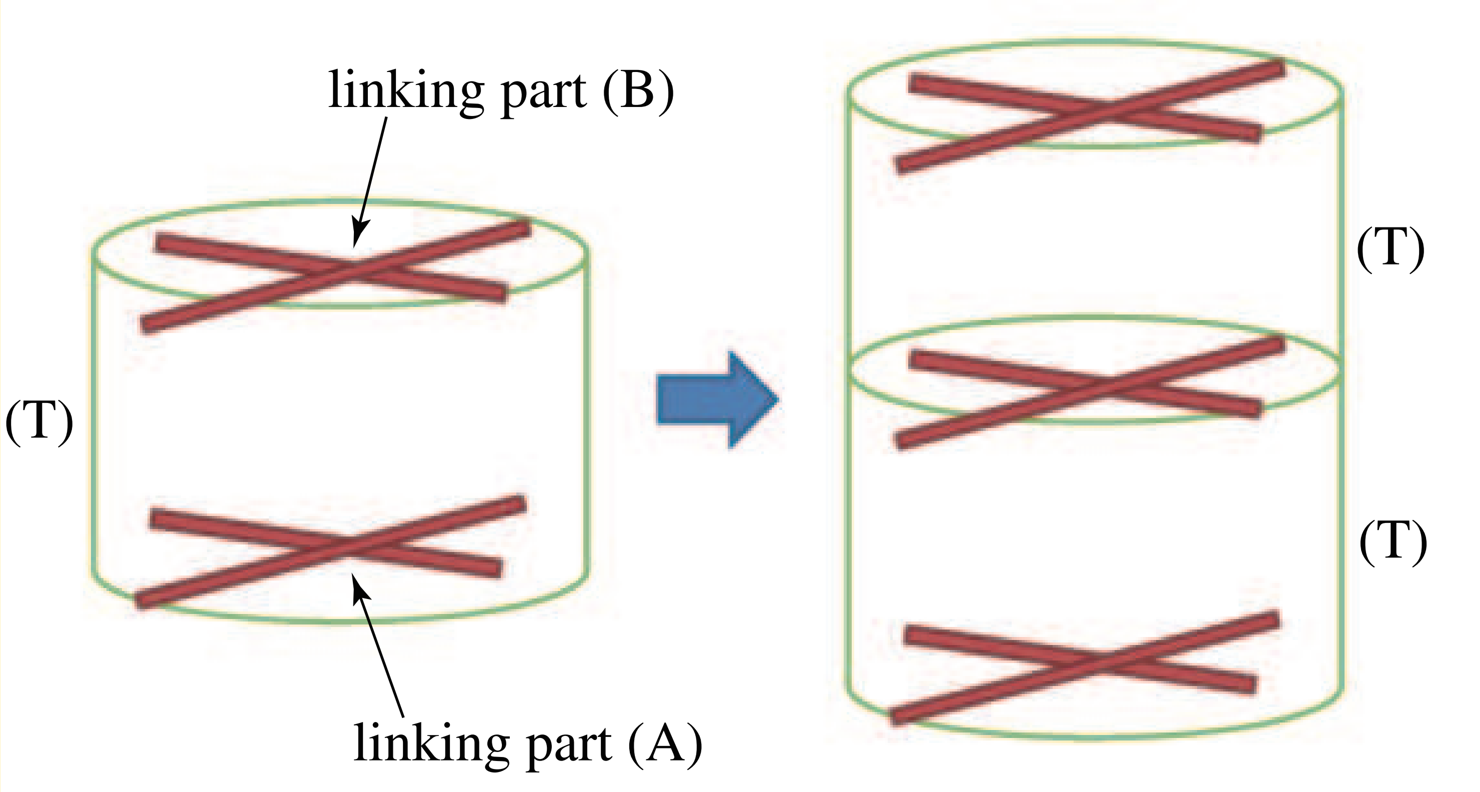


ex.) symmetric configuration

- 10 nodes
- 41 members
  (perfect graph)
- CPLEX (ver. 11.2)
- 5 struts
- solution of MIP-2
  $\longrightarrow$ 16 cables
  kinematically & statically
  indeterminate
  (prestress stable)



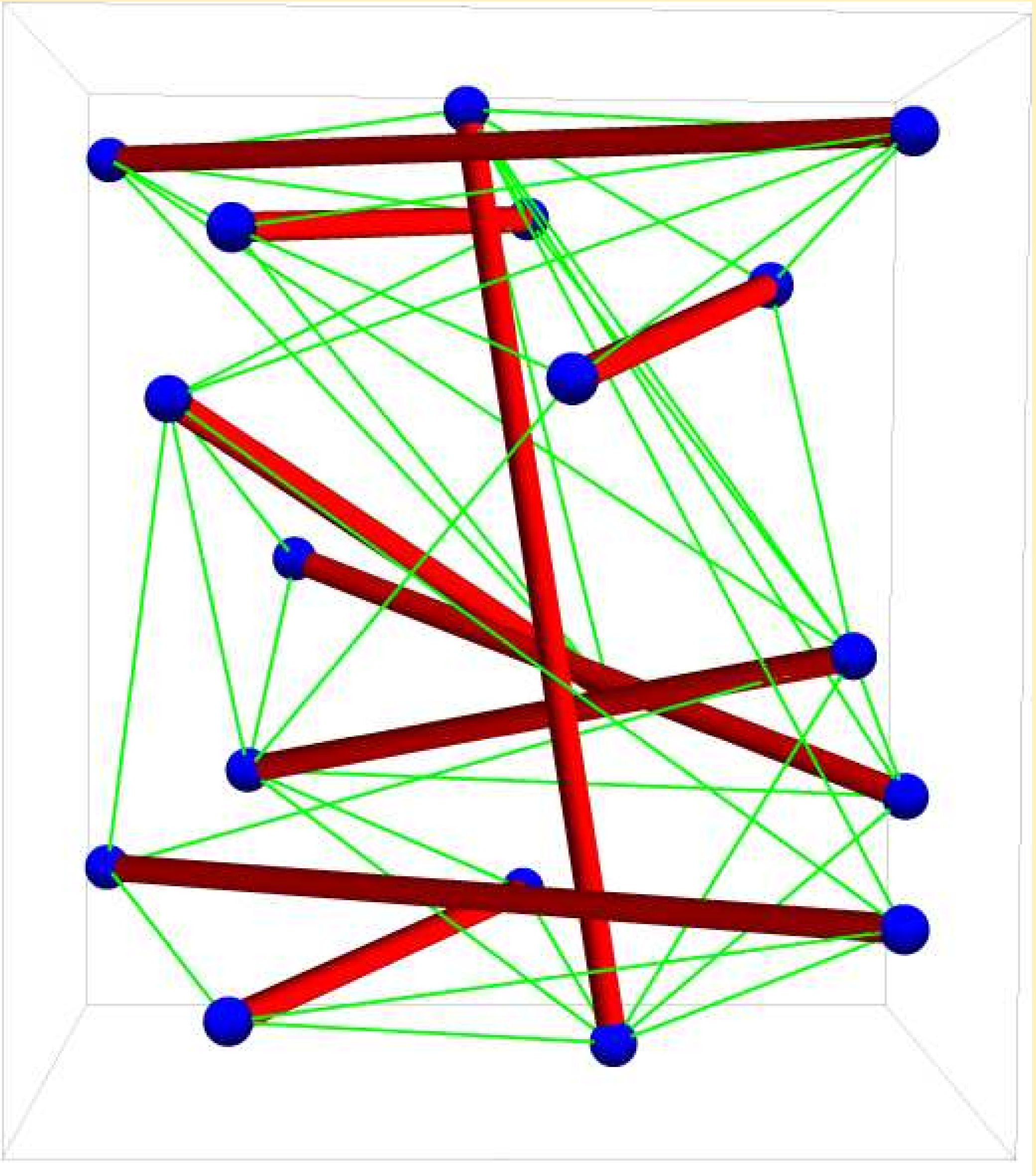
tensegrity module



- tensegrity module (T)
- "struts of (A)" are parallel to "struts of (B)"
- merge duplicate linking parts $\Rightarrow$ (T)(T)$\cdots$(T) : tensegrity

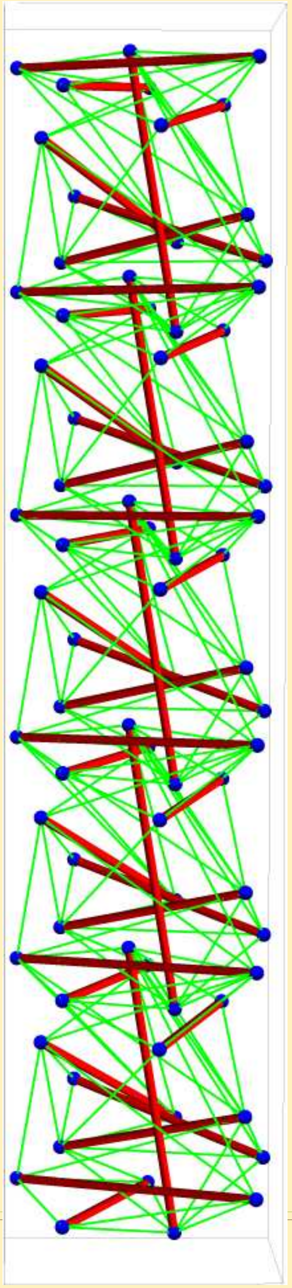
ex.) tower-type module

- 18 nodes
- perfect graph
  $\longrightarrow$ 9 struts



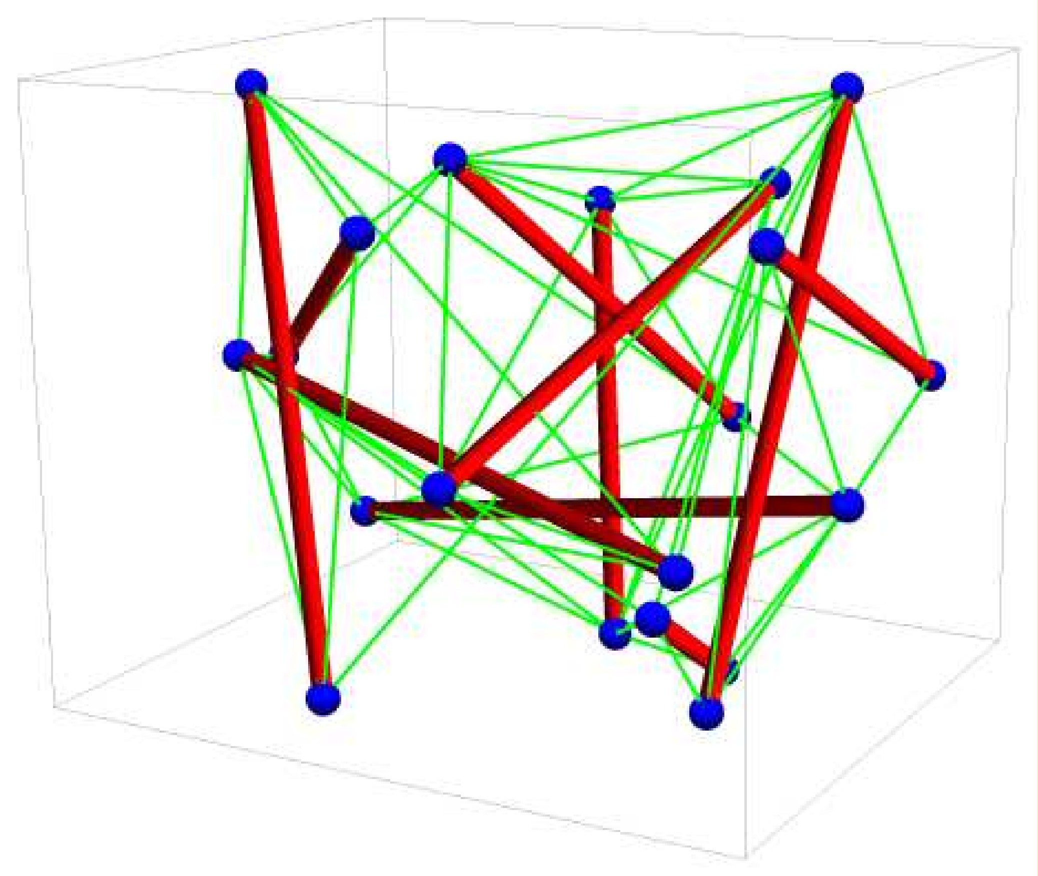
ex.) tower-type module

- 18 nodes
- perfect graph
  → 9 struts
- 5 modules
  discontinuity condition
  is still satisfied



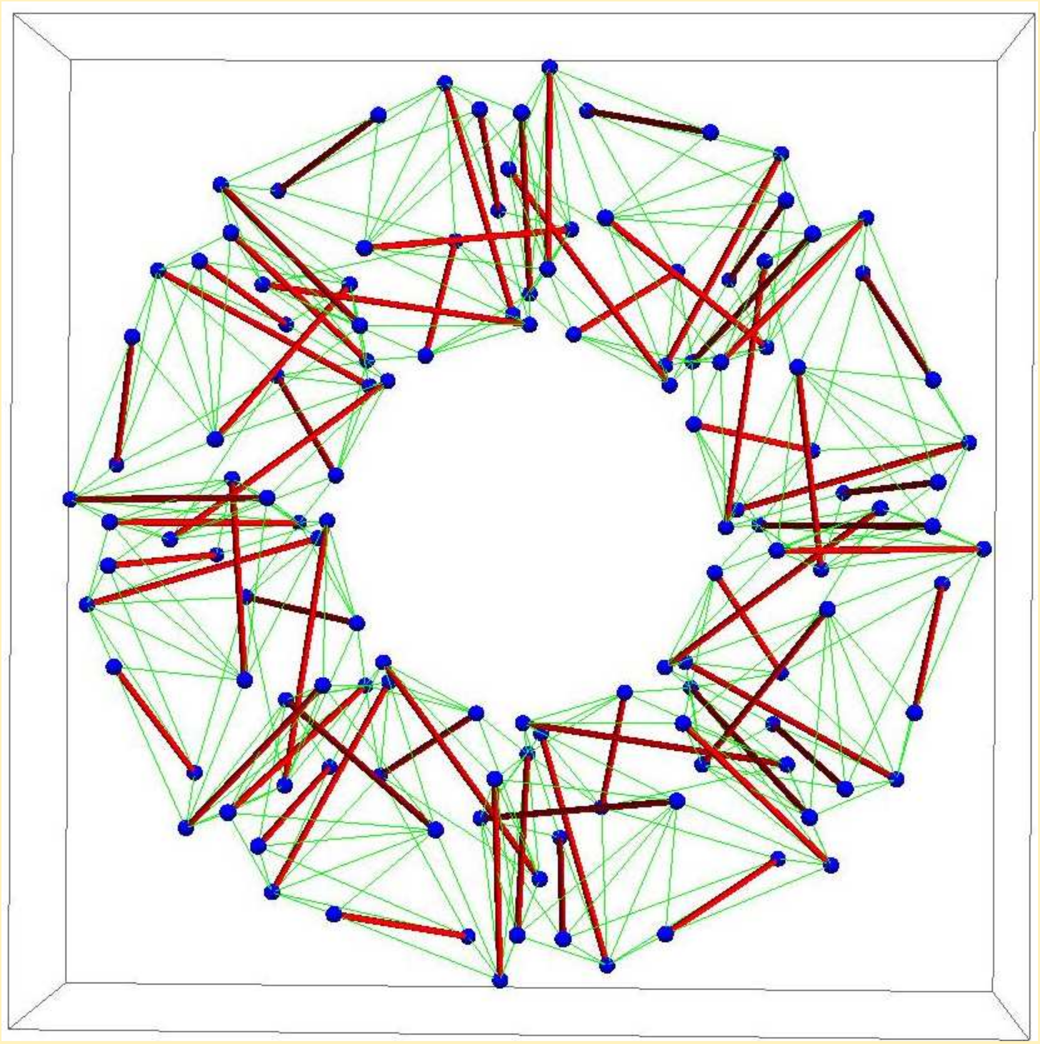
ex.) ring-type module

- 18 nodes
- perfect graph
  $\longrightarrow$ 9 struts



ex.) ring-type module

- 18 nodes
- perfect graph
  $\longrightarrow$ 9 struts
- 8 modules
  discontinuity condition
  is still satisfied



conclusions

- tensegrity
  - self-equilibrium configuration
  - discontinuous condition of struts
  - topology — connectivity of “cables & struts”
- two-stage algorithm
  - ground structure method
  - Mixed Integer Programming
    - MIP-1: max “# of struts”
    - MIP-2: min “# of cables”
  - requires no information of topology in advance