A numerical algorithm for enumerating all wedged configurations in contact problem with Coulomb friction

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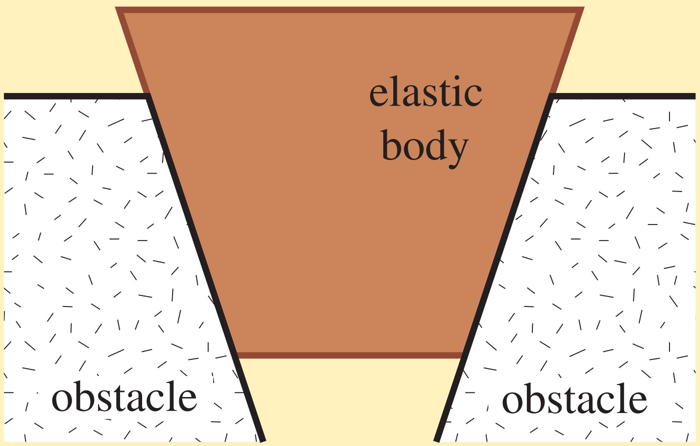
May 17, 2010

wedging problem

- linear elastic body & rigid obstacle
- Coulomb friction

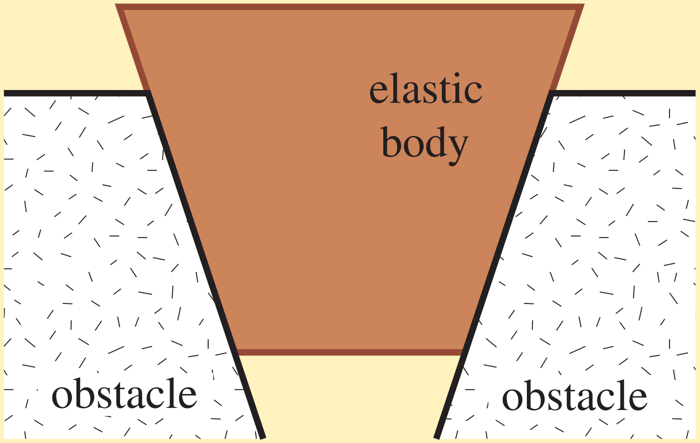


no external force
no internal force
no reaction

- cork/cap for a bottle, wedge (shim), etc.

wedging problem

- linear elastic body & rigid obstacle
- Coulomb friction



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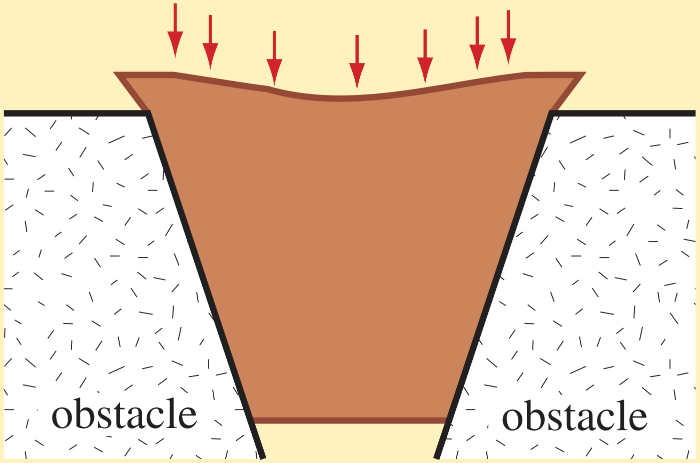
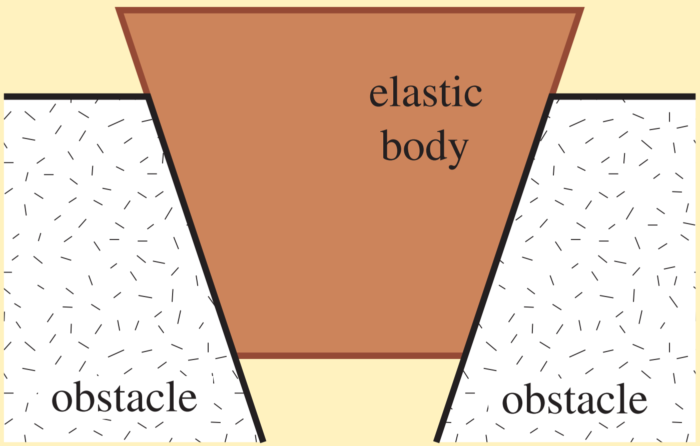


once apply external force
$\longrightarrow$ remove

wedging problem

- linear elastic body & rigid obstacle
- Coulomb friction



no external force
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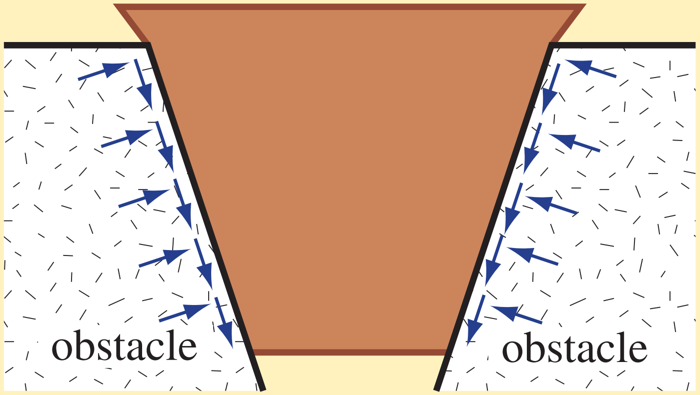
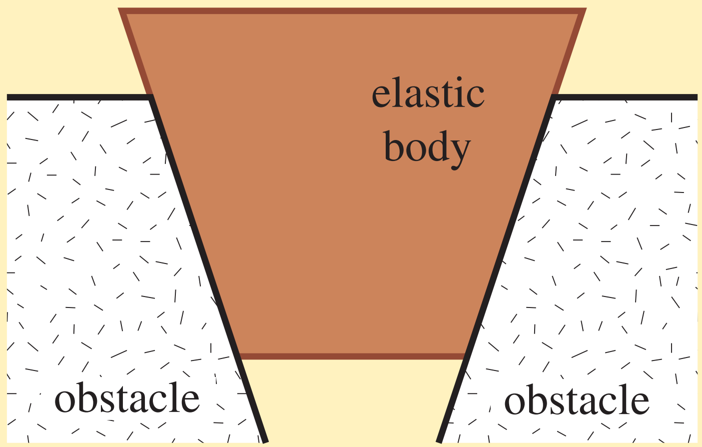


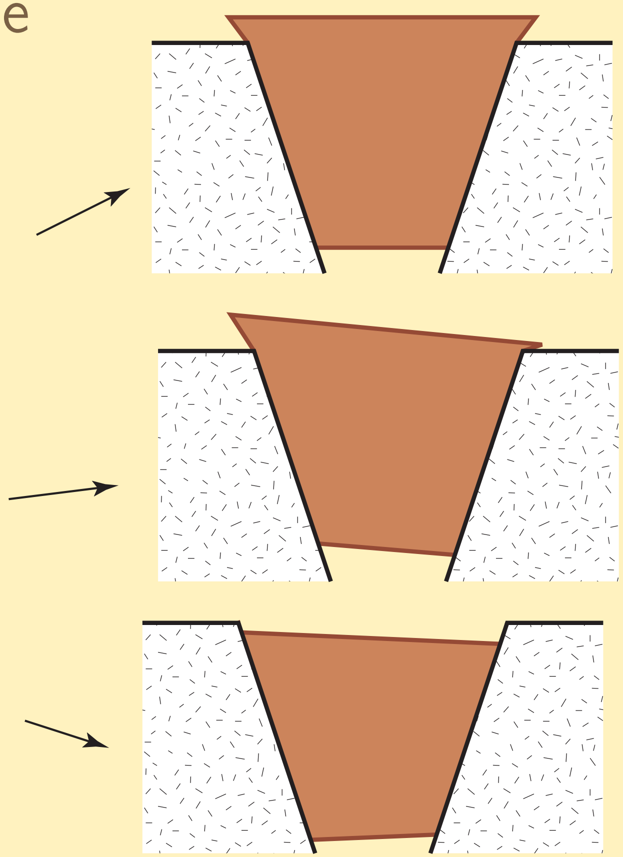
in equilibrium
without external force, but,
with internal force
and reaction

= wedged configuration

wedging problem

- linear elastic body & rigid obstacle
- Coulomb friction

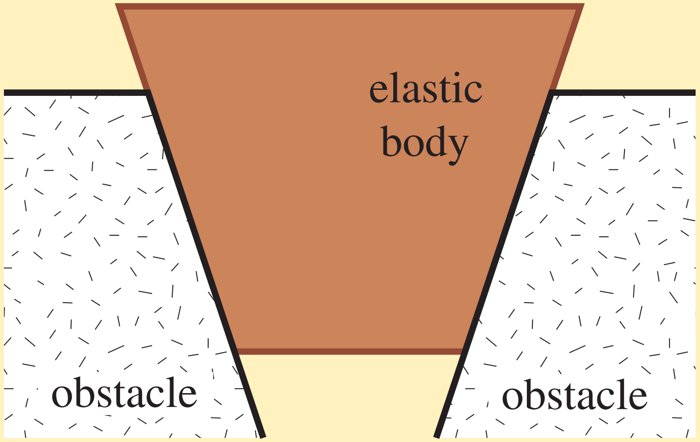


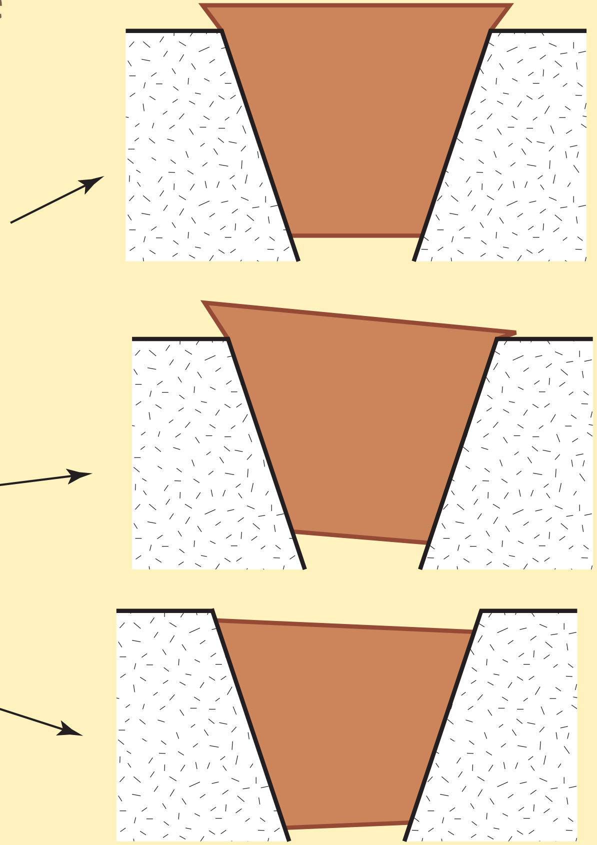


- wedged configuration is not unique
- (!) we do not specify the loading history,
  but consider only the final (possible) equilibrium configurations

wedging problem

- linear elastic body & rigid obstacle
- Coulomb friction





- wedged configuration is not unique
- $\longrightarrow$ enumerate all the wedged configurations

existing results

- wedging problem (WP)
  - [Barber & Hild 04, 06]   nonlinear eigenvalue problem
  - [Hassani, Ionescu & Oudet 07]
    - existence of the minimum value of friction coefficient μ^{c}
    - a genetic algorithm for finding (an upper bound of) μ^{c}

- related topics
  - multiplicity of solutions to quasistatic problem
    [Klarbring 90, 99]
  - enumeration of solutions to rate problem
    [Pinto da Costa & Martins 03]

existing results

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    - existence of the minimum value of friction coefficient μ^{c}
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- aim of this presentation
  - enumerate the solutions of (WP)
  - find μ^{c}

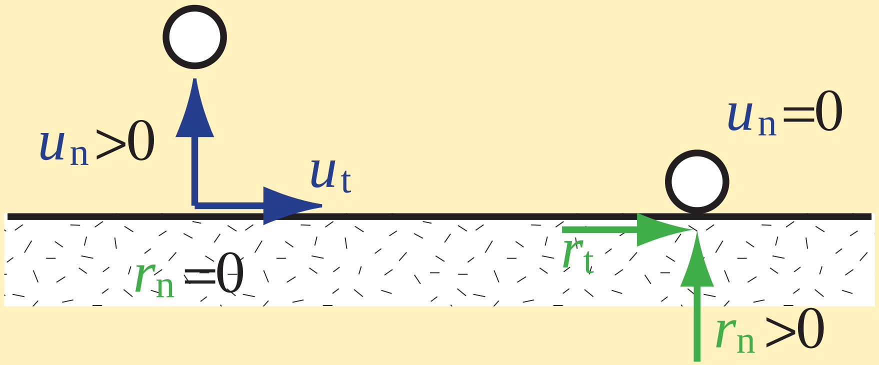
existing results

- wedging problem (WP)
  - [Barber & Hild 04, 06]   nonlinear eigenvalue problem
  - [Hassani, Ionescu & Oudet 07]
    - existence of the minimum value of friction coefficient μ^{c}
    - a genetic algorithm for finding (an upper bound of) μ^{c}
- fundamental tools
  - double description method
    [Motzkin, Raiffa, Thompson & Thrall 53]
  - enumeration algorithm for linear complementarity problem
    [de Moor, Vandenberghe & Vandewalle 92]

def. of (WP)

$K\boldsymbol{u} = \boldsymbol{r}$	(equilibrium eq.)
$\mu r_{\mathrm{n}i} \geq \|\boldsymbol{r}_{\mathrm{t}i}\| \quad (i = 1, \dots, m)$	(friction cones)
$\boldsymbol{u}_{\mathrm{n}} \geq \mathbf{0}, \quad \boldsymbol{r}_{\mathrm{n}} \geq \mathbf{0}, \quad {\boldsymbol{u}_{\mathrm{n}}}^{\mathrm{T}} \boldsymbol{r}_{\mathrm{n}} = 0$	(unilateral condn.)

- given:
  - $K \in \mathbb{R}^{m \times m}$: stiffness matrix
    m : # of contact candidate nodes
  - $\mu > 0$: friction coefficient



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  - $K \in \mathbb{R}^{m \times m}$: stiffness matrix
    m : # of contact candidate nodes
  - $\mu > 0$: friction coefficient
- find: $(\boldsymbol{u}, \boldsymbol{r}) \neq \boldsymbol{0}$
  - $\boldsymbol{u} = (\boldsymbol{u}_{\mathrm{n}}, \boldsymbol{u}_{\mathrm{t}})$: displacements
  - $\boldsymbol{r} = (\boldsymbol{r}_{\mathrm{n}}, \boldsymbol{r}_{\mathrm{t}})$: reactions

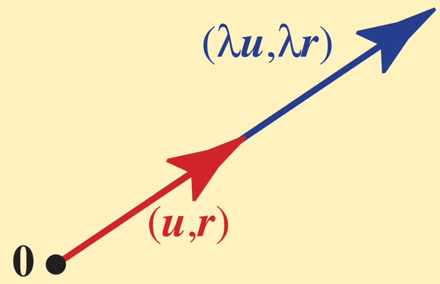
def. of (WP)

$$K\boldsymbol{u} = \boldsymbol{r} \qquad \text{(equilibrium eq.)}$$

$$\mu r_{\mathrm{n}i} \geq \|\boldsymbol{r}_{\mathrm{t}i}\| \quad (i = 1, \ldots, m) \qquad \text{(friction cones)}$$

$$\boldsymbol{u}_{\mathrm{n}} \geq \boldsymbol{0}, \quad \boldsymbol{r}_{\mathrm{n}} \geq \boldsymbol{0}, \quad {\boldsymbol{u}_{\mathrm{n}}}^{\mathrm{T}} \boldsymbol{r}_{\mathrm{n}} = 0 \qquad \text{(unilateral condn.)}$$

- property: If $(\boldsymbol{u}, \boldsymbol{r})$ is a solution of (WP),
  - $\Rightarrow \ \forall \lambda > 0 \ :$ $(\lambda \boldsymbol{u}, \lambda \boldsymbol{r})$ is a solution of (WP)
  - $\Rightarrow \ \forall \mu' > \mu \ :$ $(\boldsymbol{u}, \boldsymbol{r})$ is a solution for (WP)



def. of (WP)

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  - $\Rightarrow \ \forall \mu' > \mu \ : \ (\boldsymbol{u}, \boldsymbol{r})$ is a solution for (WP)
- $\exists \mu^{\mathrm{c}} > 0$
  s.t. (WP) has a solution for any $\mu \geq \mu^{\mathrm{c}}$

[Hassani, Ionescu & Oudet 07]

(WP) in 2D

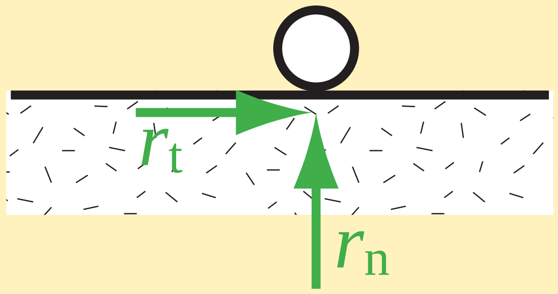
$$K\boldsymbol{u} = \boldsymbol{r}$$

$$-\mu r_{\mathrm{n}i} \leq r_{\mathrm{t}i} \leq \mu r_{\mathrm{n}i}$$

$$\boldsymbol{u}_{\mathrm{n}} \geq \boldsymbol{0}, \quad \boldsymbol{r}_{\mathrm{n}} \geq \boldsymbol{0}$$

$$\boldsymbol{u}_{\mathrm{n}}{}^{\mathrm{T}} \boldsymbol{r}_{\mathrm{n}} = 0$$

- $r_{\mathrm{n}i}, r_{\mathrm{t}i} \in \mathbb{R} \quad \rightarrow$
  - friction cone, "$\mu r_{\mathrm{n}i} \geq |r_{\mathrm{t}i}|$", is reduced to linear inequalities



(WP) in 2D

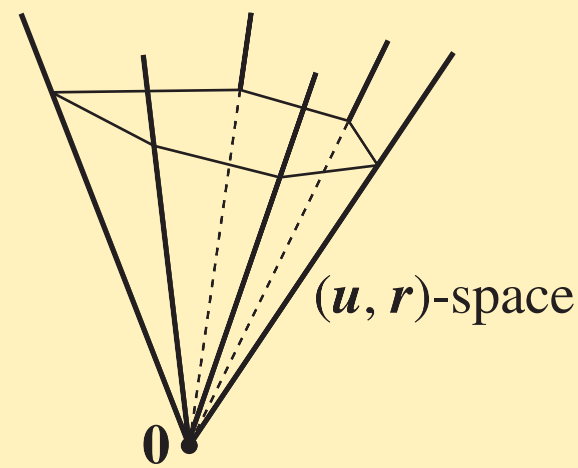
$$
\left.
\begin{aligned}
& K\boldsymbol{u} = \boldsymbol{r} \\
& -\mu r_{\mathrm{n}i} \le r_{\mathrm{t}i} \le \mu r_{\mathrm{n}i} \\
& \boldsymbol{u}_{\mathrm{n}} \ge \mathbf{0}, \quad \boldsymbol{r}_{\mathrm{n}} \ge \mathbf{0}
\end{aligned}
\right\} \text{ linear inequalities}
$$

$$
\boldsymbol{u}_{\mathrm{n}}{}^{\mathrm{T}} \boldsymbol{r}_{\mathrm{n}} = 0 \qquad \text{complementarity}
$$

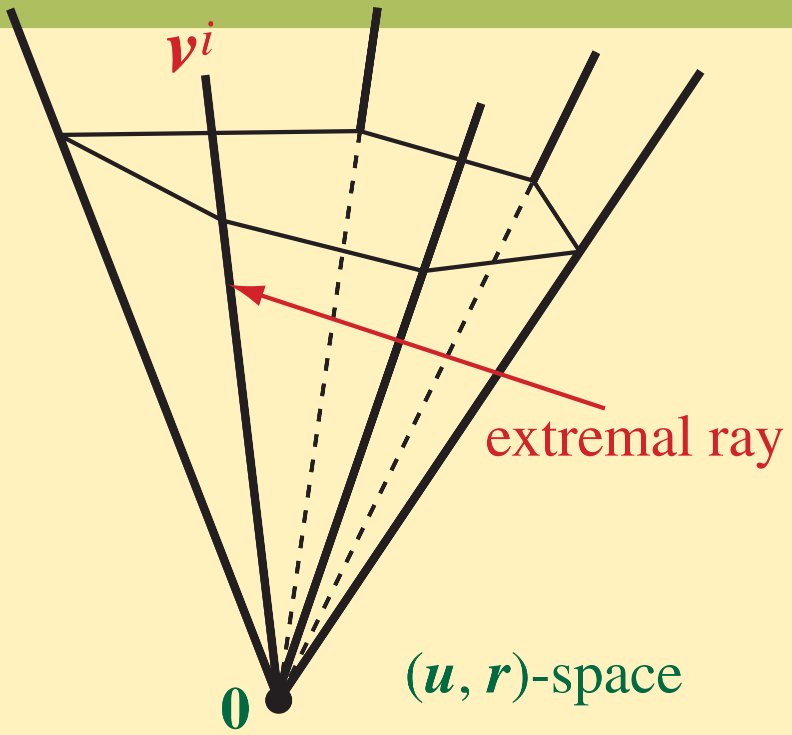
- (WP) =
  linear inequalities & complementarity conditions

  ⇕

  "polyhedral cone"



solution set of (WP) in 2D

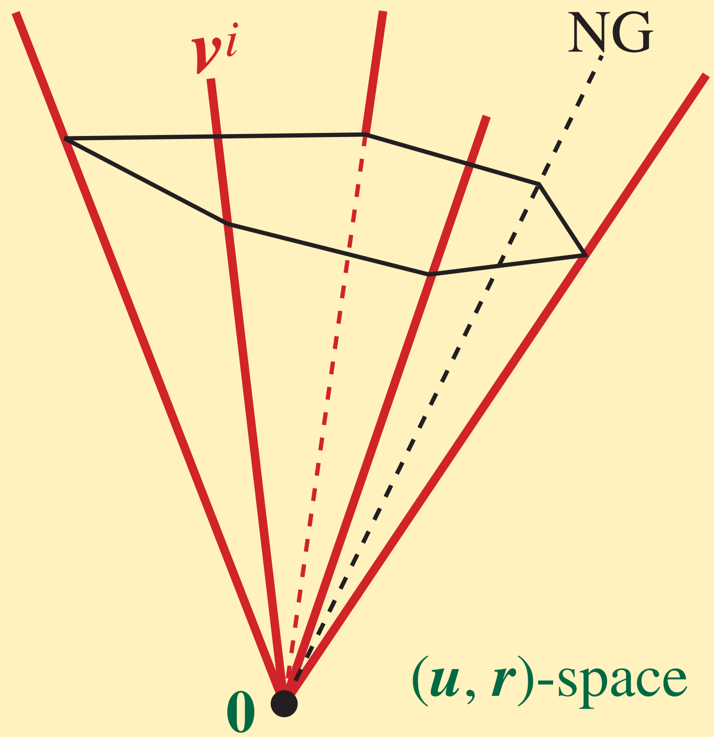


- enumerate extremal rays ($\boldsymbol{v}^i$'s) of the polyhedral cone
  - $\longrightarrow$

$$\text{cone} = \left\{ \sum_{i=1}^{\ell} \lambda_i \boldsymbol{v}^i \mid \lambda_i \geq 0 \ (\forall i) \right\}$$

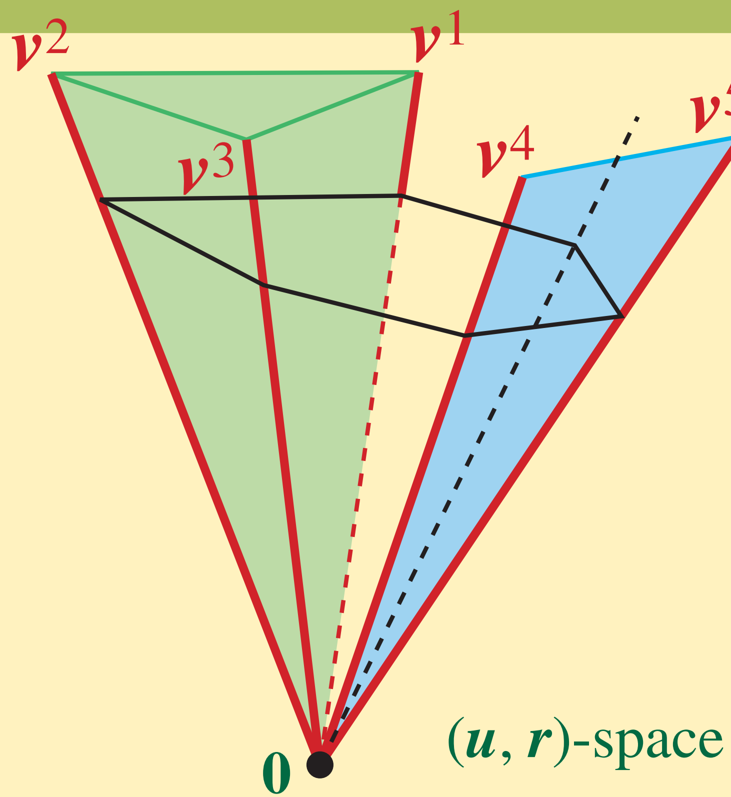
(nonnegative combination of $\boldsymbol{v}^i$)

solution set of (WP) in 2D



- enumerate extremal rays ($\boldsymbol{v}^i$'s) of the polyhedral cone
- remove extremal rays which do not satisfy the complementarity conditions
  - check

$$(\boldsymbol{u}_{\mathrm{n}}^i)^{\mathrm{T}} \boldsymbol{r}_{\mathrm{n}}^i \begin{cases} = 0 \\ \neq 0 \end{cases}$$

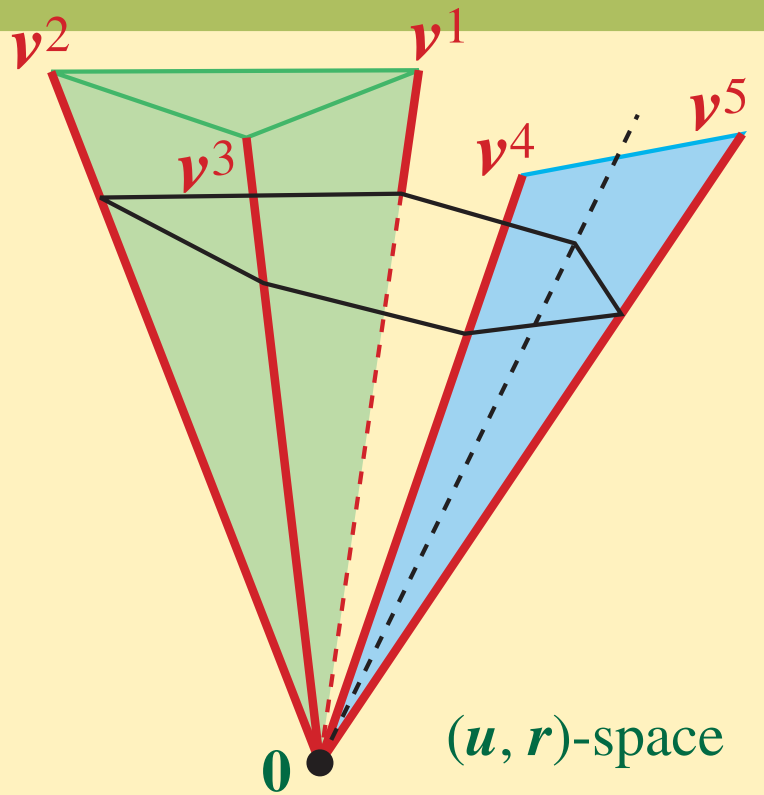


	v^1	v^2	v^3	v^4	v^5
u_{n1}	0	0	0	+	+
r_{n1}	+	+	+	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{nm}	+	0	+	0	0
r_{nm}	0	0	0	+	+

- enumerate extremal rays ($\boldsymbol{v}^i$'s) of the polyhedral cone
- remove extremal rays which do not satisfy the complementarity conditions
- find sets of $\boldsymbol{v}^i$'s satisfying the *cross-complementarity condition*, say

$$\Big(\sum_{i \in B} \boldsymbol{u}_{\mathrm{n}}^i\Big)^{\mathrm{T}} \Big(\sum_{i \in B} \boldsymbol{r}_{\mathrm{n}}^i\Big) = 0.$$

solution set of (WP) in 2D

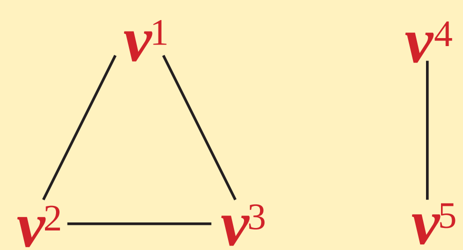


	$\boldsymbol{v}^1$	$\boldsymbol{v}^2$	$\boldsymbol{v}^3$	$\boldsymbol{v}^4$	$\boldsymbol{v}^5$
$u_{\mathrm{n}1}$	0	0	0	+	+
$r_{\mathrm{n}1}$	+	+	+	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{\mathrm{n}m}$	+	0	+	0	0
$r_{\mathrm{n}m}$	0	0	0	+	+

- enumerate extremal rays ($\boldsymbol{v}^i$'s) of the polyhedral cone
- remove extremal rays which do not satisfy the complementarity conditions
- find sets of $\boldsymbol{v}^i$'s satisfying the *cross-complementarity condition*.

$\Updownarrow$

find the maximal cliques of the graph:



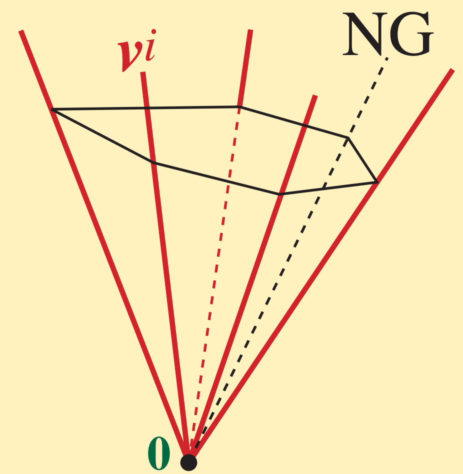
enumeration of solutions to (WP)

- 1) find all the extremal rays ($\boldsymbol{v}^i$'s) of the polyhedral cone
  — double description method [Motzkin *et al.* 53]

- 2) remove extremal rays which do not satisfy
  the complementarity conditions

- 3) find cross-complementarity relation of $\boldsymbol{v}^i$'s
  — finding maximal cliques → e.g., [Makino & Uno 04]

↓

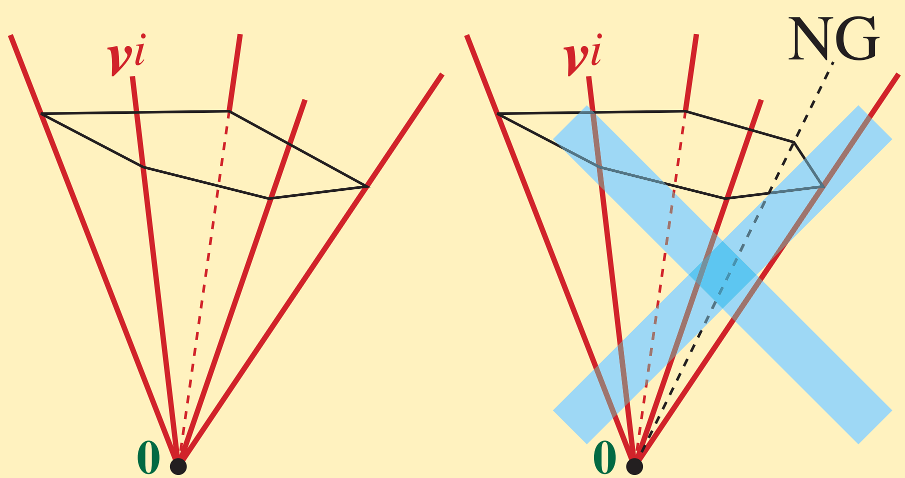


enumeration of solutions to (WP)

- 1) find all the extremal rays ($\boldsymbol{v}^i$'s) of the polyhedral cone
  — double description method [Motzkin *et al.* 53]
- 2) remove extremal rays which do not satisfy
  the complementarity conditions
- 3) find cross-complementarity relation of $\boldsymbol{v}^i$'s
  — finding maximal cliques → e.g., [Makino & Uno 04]

↓

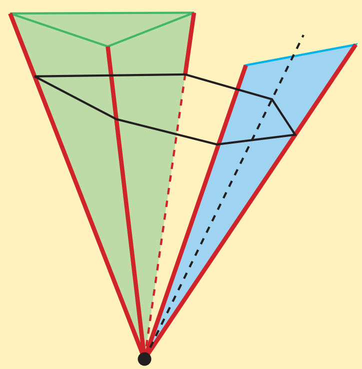
- 1) + 2) →
  enumerate only the extremal rays satisfying the complementarity conditions



critical friction coefficient μ^{c}

- for any $\mu \geq \mu^{\mathrm{c}}$, (WP) has a solution ($\forall \mu < \mu^{\mathrm{c}}$, no solution)

[Hassani *et al.* 07]



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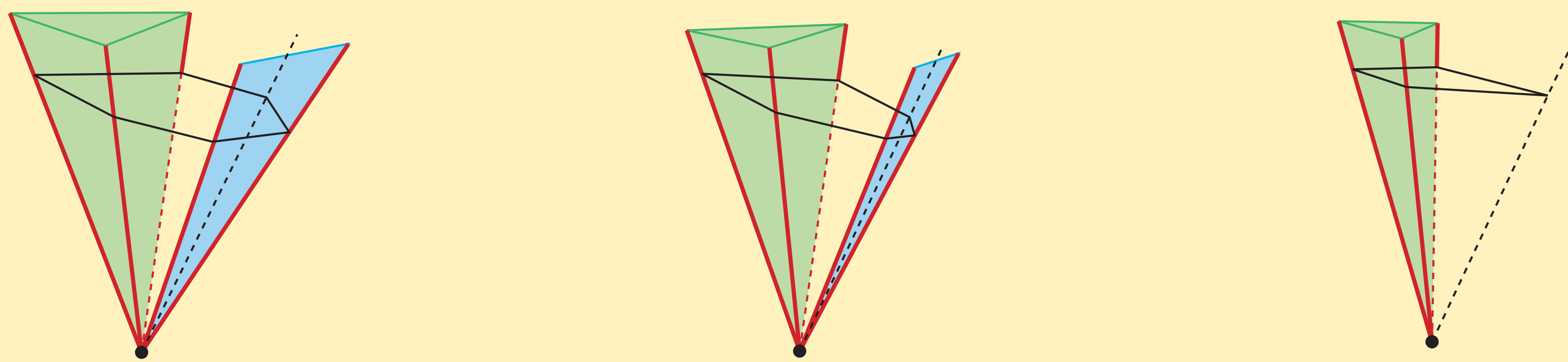

μ : large $\leftrightarrow$ μ : small

- for a fixed μ, check each polyhedral cone is empty or not
  $\Leftrightarrow$ solve a convex optimization problem

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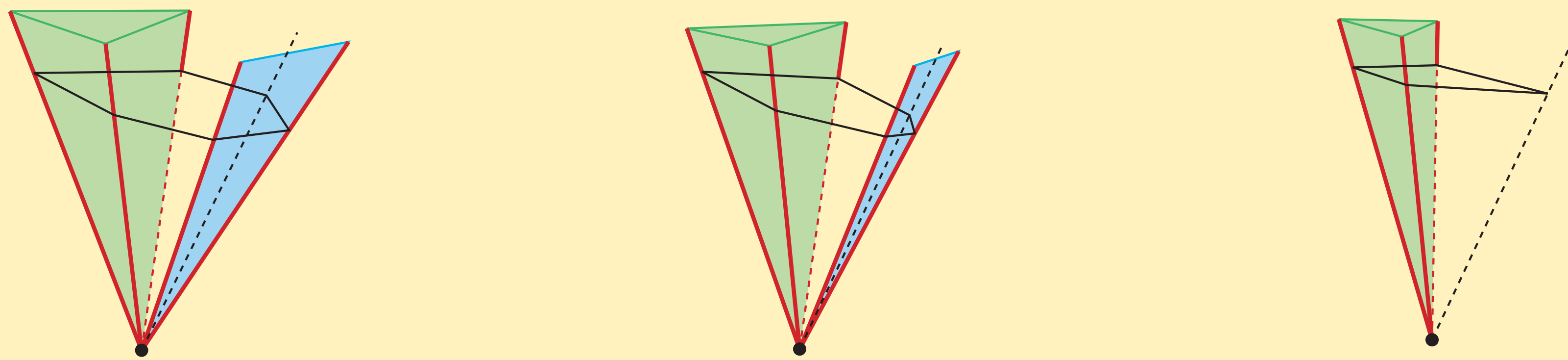


μ : large $\leftrightarrow$ μ : small

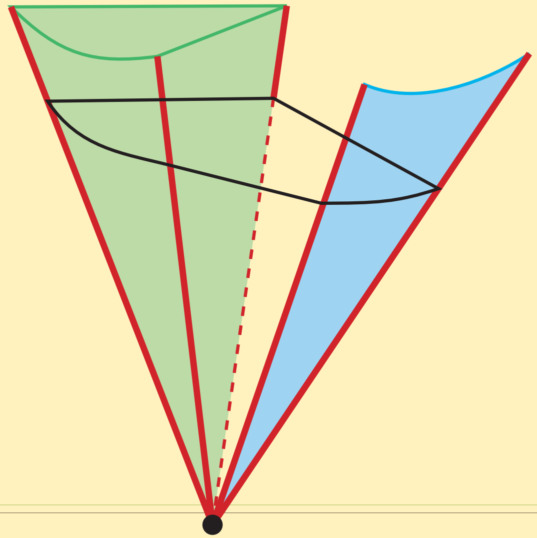
- for a fixed μ, check each polyhedral cone is empty or not
  $\Leftrightarrow$ solve a convex optimization problem
- bisection method with respect to μ
  $\rightarrow \mu^{c}$ for each cone is obtained
- minimum among these μ^{c}'s — critical value

solution set of (WP) in 3D

$$\left.\begin{array}{l} K\boldsymbol{u} = \boldsymbol{r} \\ \mu r_{\mathrm{n}i} \geq \left\| \begin{bmatrix} r_{\mathrm{t}i} \\ r_{\mathrm{o}i} \end{bmatrix} \right\| \\ \boldsymbol{u}_{\mathrm{n}} \geq \boldsymbol{0}, \quad \boldsymbol{r}_{\mathrm{n}} \geq \boldsymbol{0} \end{array}\right\} \quad \text{cone (not polyhedral)}$$

$$\boldsymbol{u}_{\mathrm{n}}{}^{\mathrm{T}} \boldsymbol{r}_{\mathrm{n}} = 0 \qquad \text{complementarity}$$

- solution set $= \bigcup$ convex cones
  — cannot be described by finitely many representative solutions



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- solution set $= \bigcup$ convex cones
  — cannot be described by finitely many representative solutions
- enumeration of sign-patterns of $\boldsymbol{u}_{\mathrm{n}}$ & $\boldsymbol{r}_{\mathrm{n}}$

	B^1	B^2	B^3	$\cdots$
$u_{\mathrm{n}1}$	0	$+$	$+$	$\cdots$
$r_{\mathrm{n}1}$	$+$	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$		
$u_{\mathrm{n}m}$	$+$	0	0	$\cdots$
$r_{\mathrm{n}m}$	0	0	$+$	$\cdots$

solution set of (WP) in 3D

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& \boldsymbol{u}_{\mathrm{n}} \geq \boldsymbol{0}, \quad \boldsymbol{r}_{\mathrm{n}} \geq \boldsymbol{0}
\end{aligned}
\right\} \text{cone (not polyhedral)}
$$

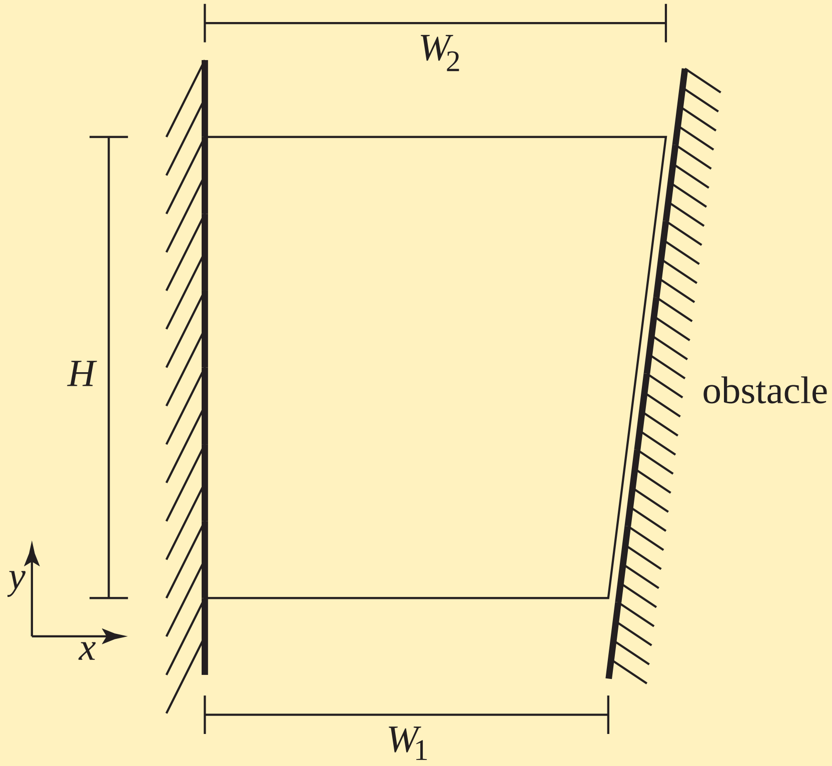
$$
\boldsymbol{u}_{\mathrm{n}}{}^{\mathrm{T}} \boldsymbol{r}_{\mathrm{n}} = 0 \qquad \text{complementarity}
$$

- solution set $= \bigcup$ convex cones
  — cannot be described by finitely many representative solutions
- enumeration of sign-patterns of $\boldsymbol{u}_{\mathrm{n}}$ & $\boldsymbol{r}_{\mathrm{n}}$
- (1) branch-and-bound method
- (2) polyhedral approximation
  + feasibility problem

	B^1	B^2	B^3	$\cdots$
$u_{\mathrm{n}1}$	0	$+$	$+$	$\cdots$
$r_{\mathrm{n}1}$	$+$	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$		
$u_{\mathrm{n}m}$	$+$	0	0	$\cdots$
$r_{\mathrm{n}m}$	0	0	$+$	$\cdots$

Ex.) body in plane stress

- 12 contact candidate nodes (FEM discretization)
- on the rigid obstacle at the initial state



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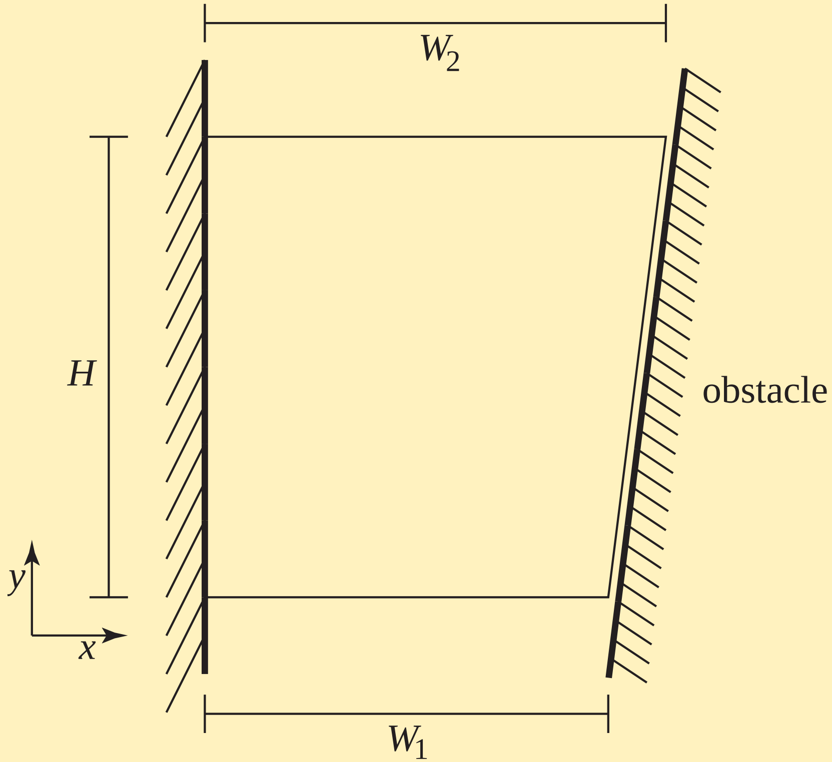
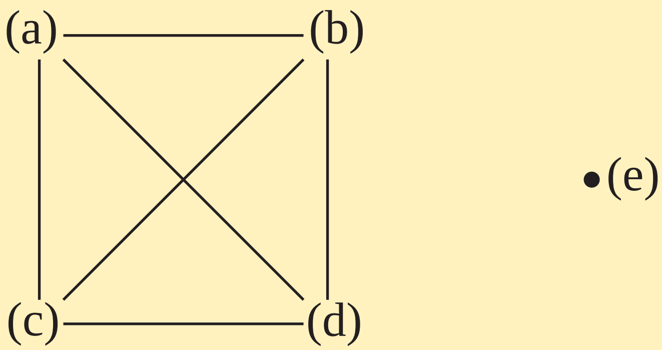
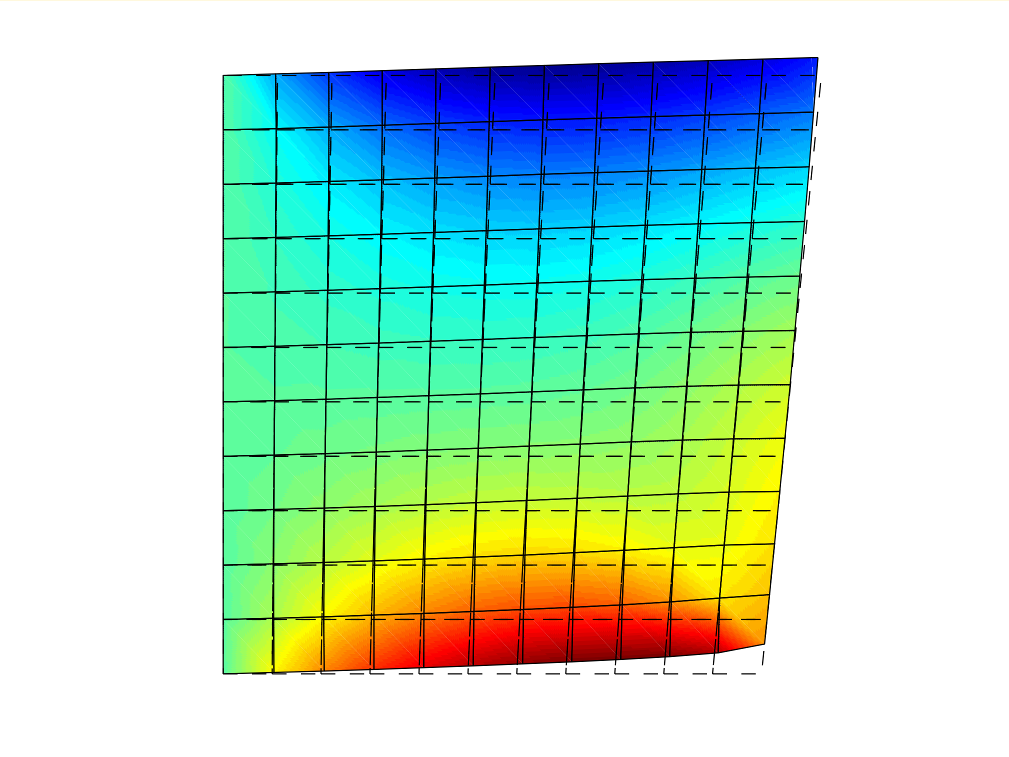




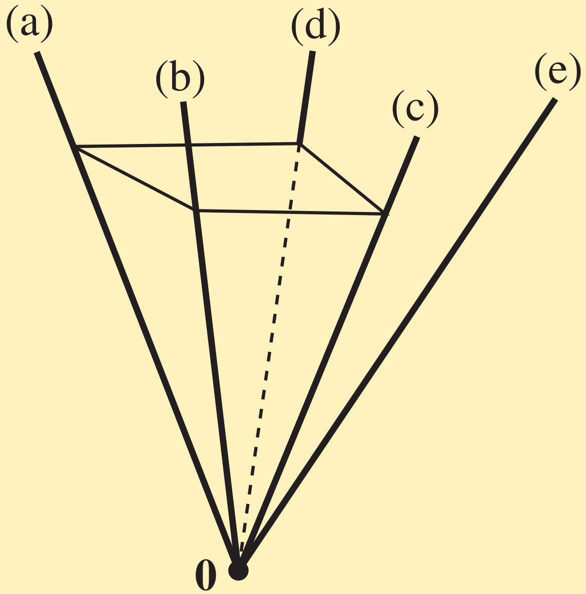
cross-complementarity relationship among 5 extremal rays

Ex.) body in plane stress

- 12 contact candidate nodes $\mu = 1.5$
- 5 extremal rays



ex. ray (a)



solution set

Ex.) body in plane stress

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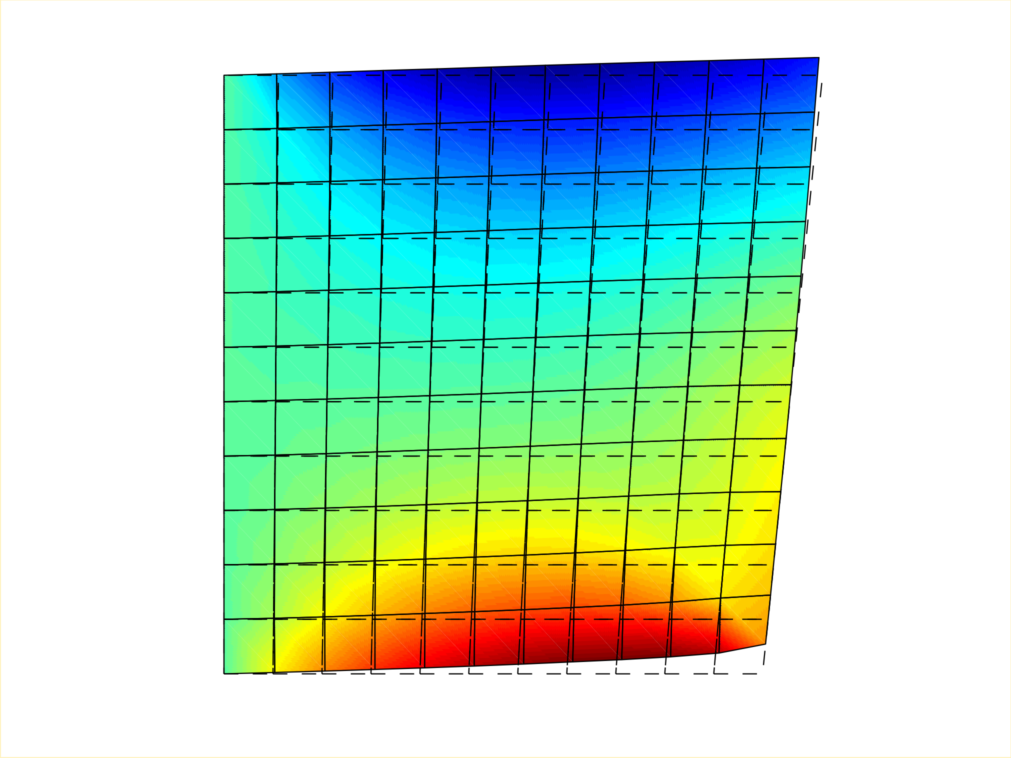
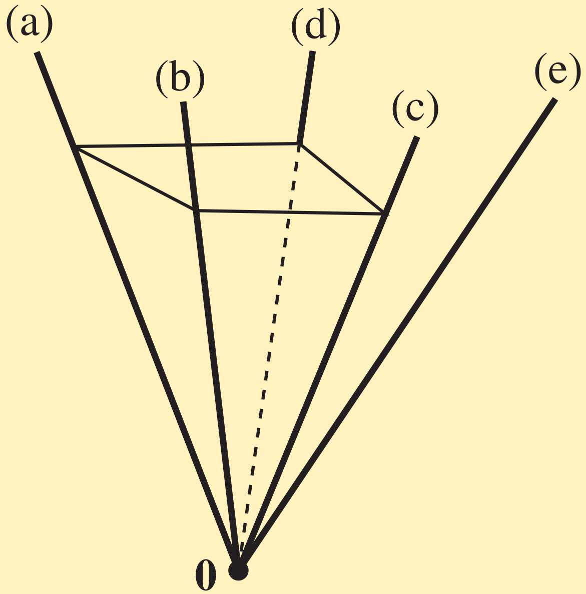


ex. ray (b)



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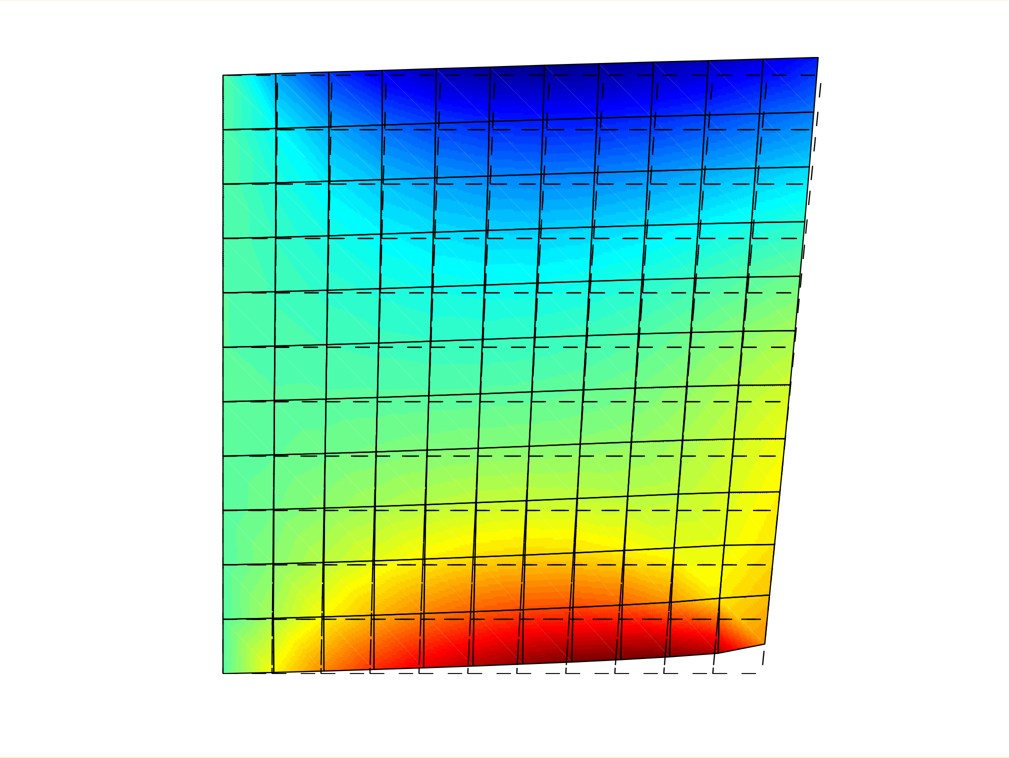
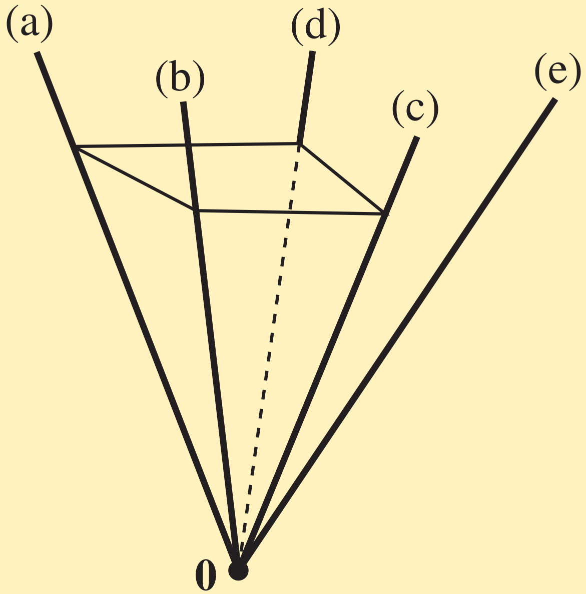


ex. ray (c)



solution set

Ex.) body in plane stress

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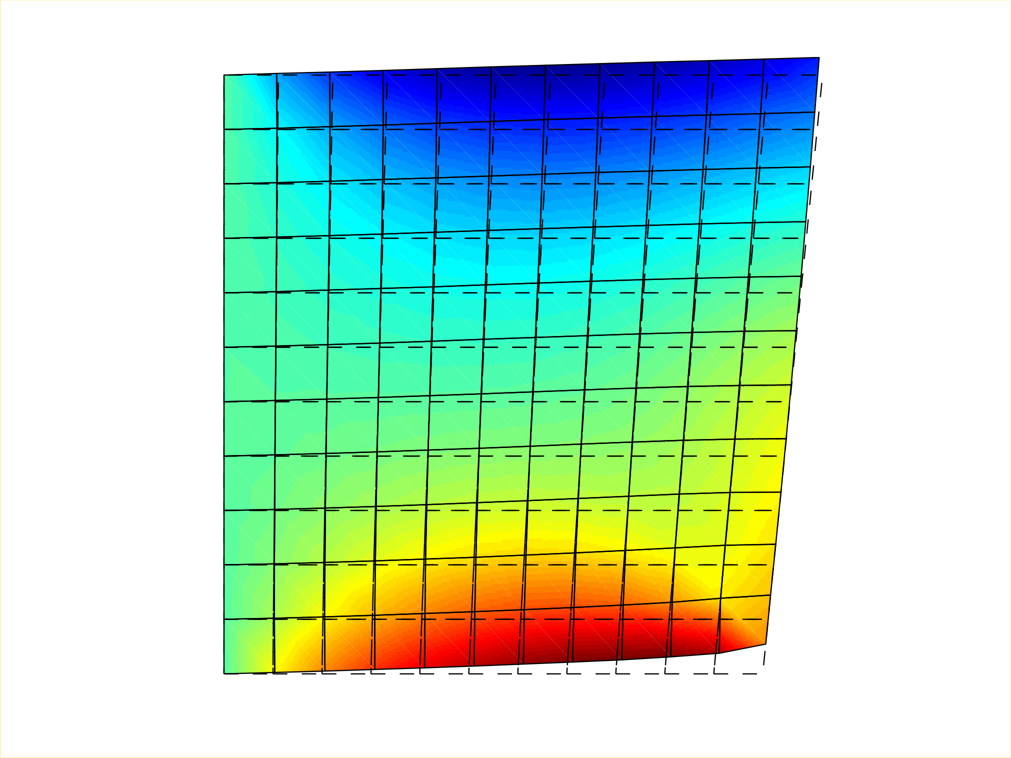
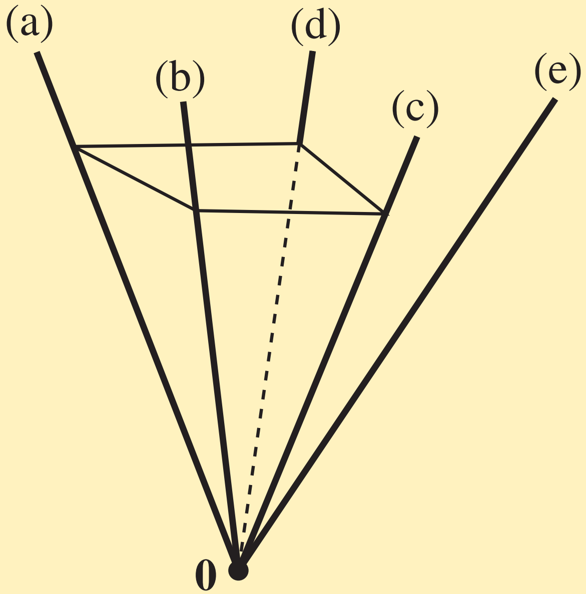
- 5 extremal rays



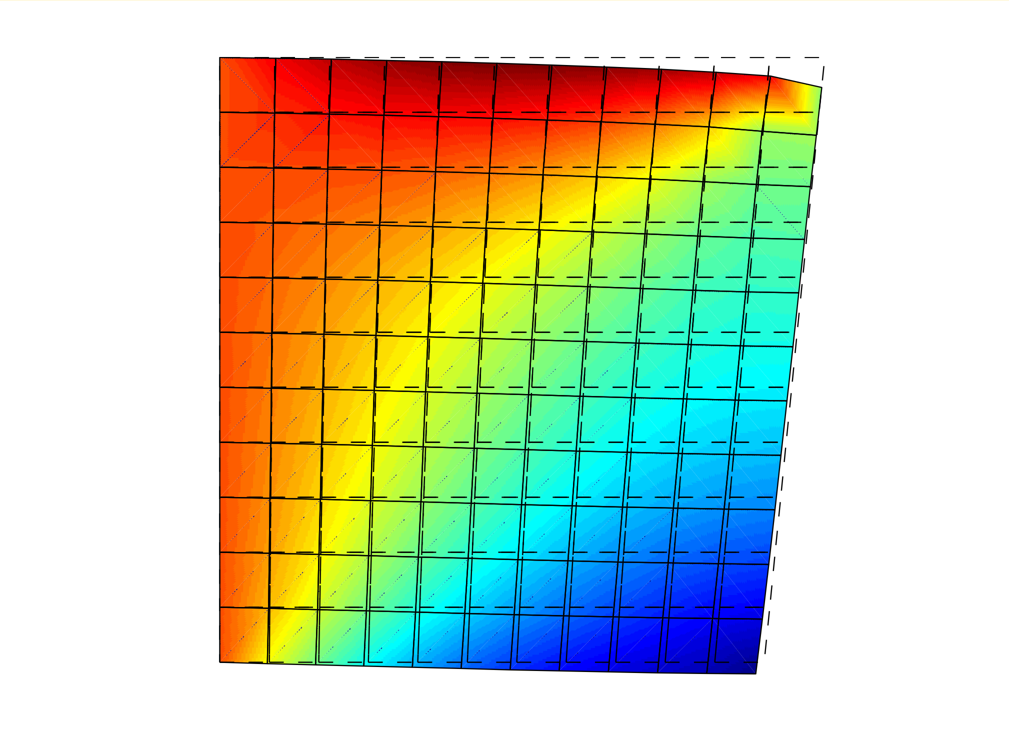
ex. ray (d)



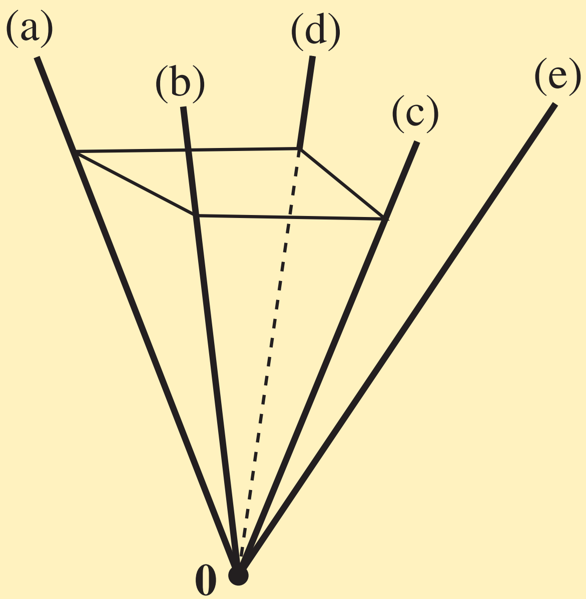
solution set

Ex.) body in plane stress

- 12 contact candidate nodes $\mu = 1.5$
- 5 extremal rays

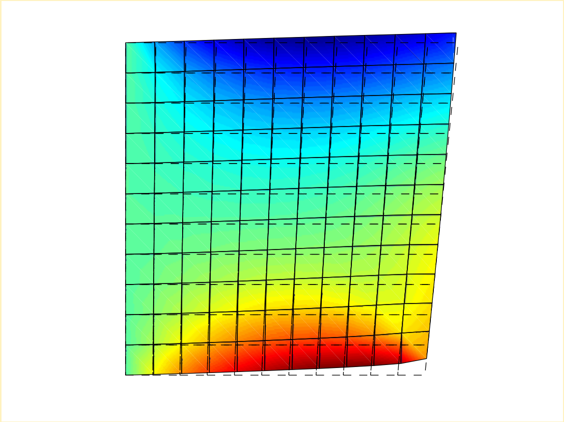


ex. ray (e)

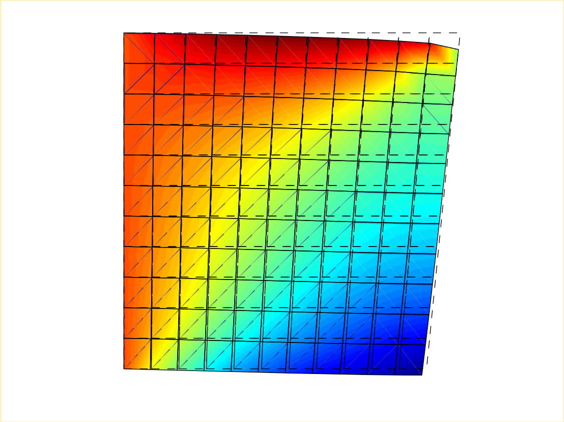


solution set

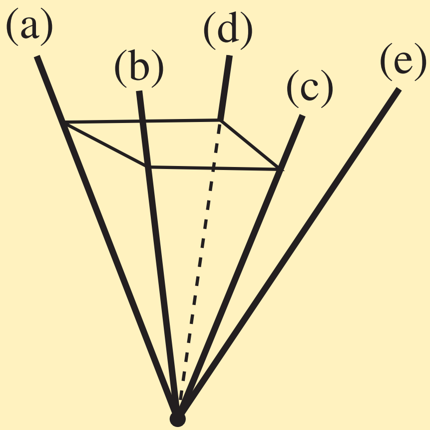
- critical value of friction coefficient: μ^{c}
- bi-section method for each polyhedral cone



ex. ray (a)

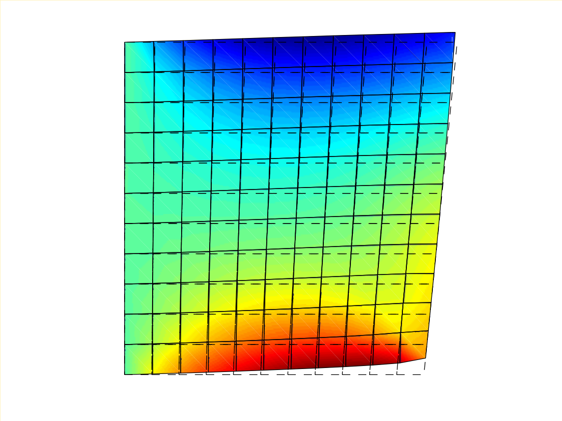


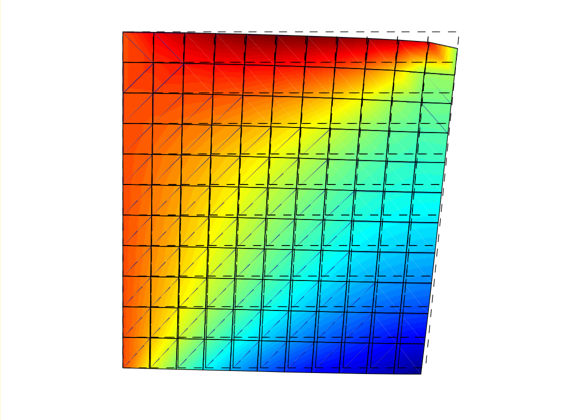
ex. ray (e)

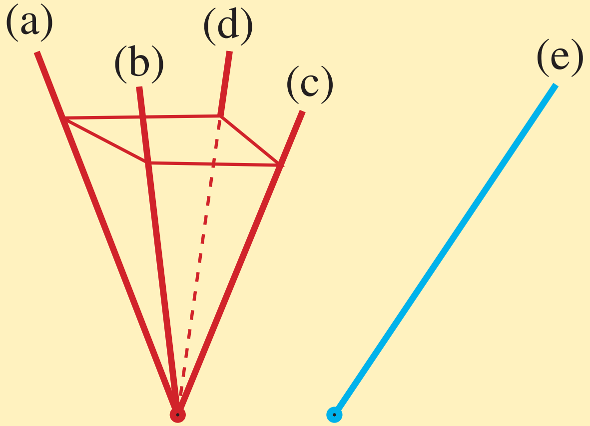


solution set

- critical value of friction coefficient: μ^c
- bi-section method for each polyhedral cone


ex. ray (a)

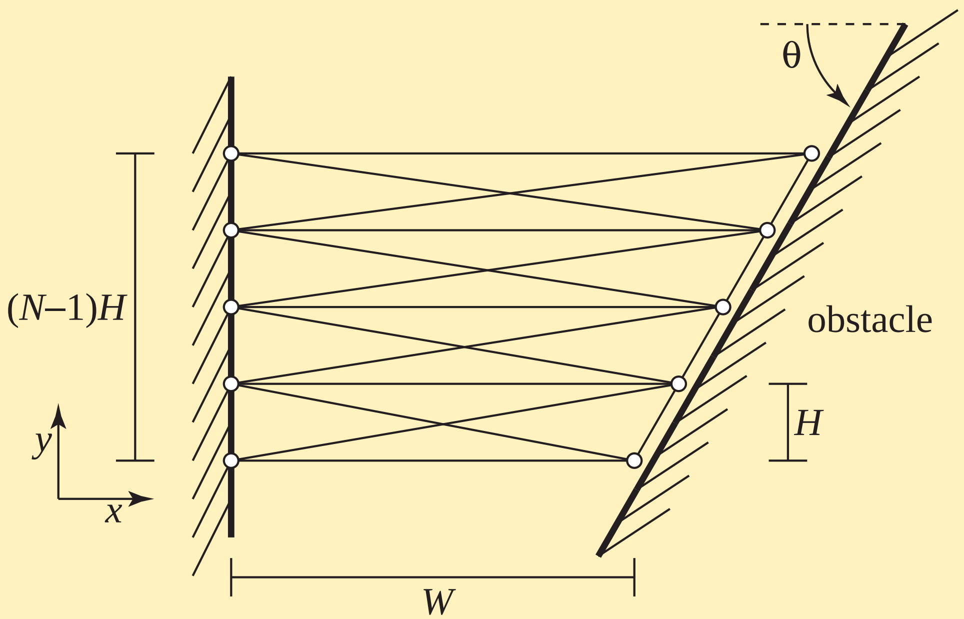

ex. ray (e)



sign-pattern	μ^c
(a)–(d)	1.4545
(e)	1.2689

Ex.) plane truss

- N contact candidate nodes
- on the rigid obstacle at the initial state

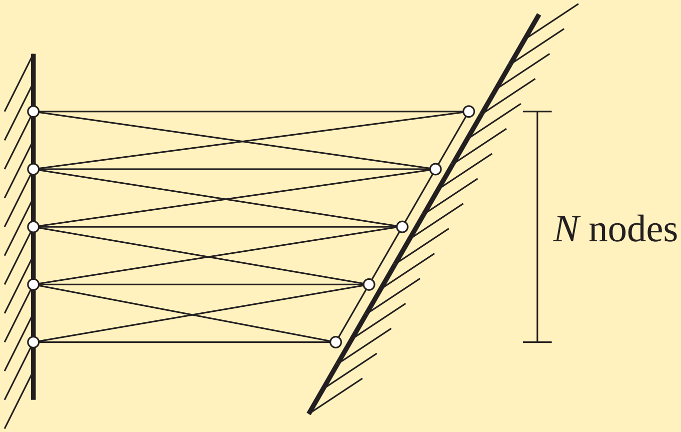


$N = 5$

Ex.) plane truss

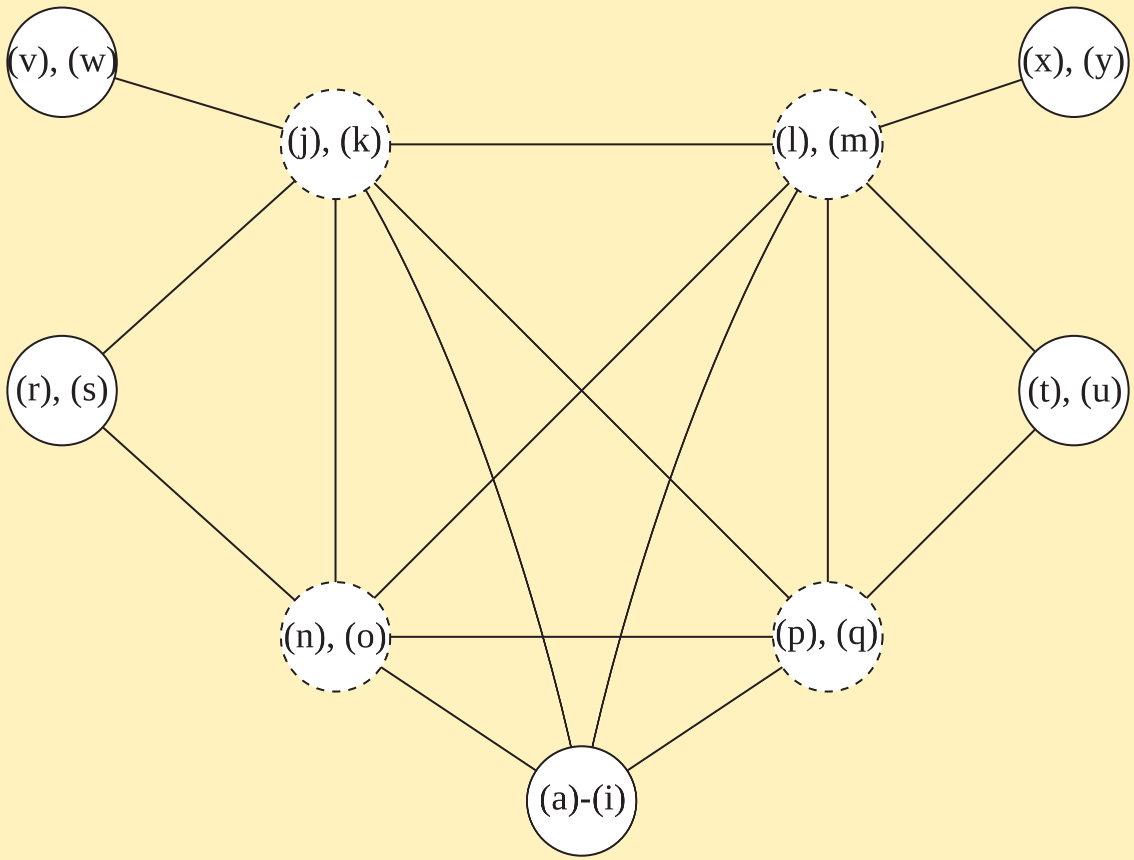
- N contact candidate nodes
- $\mu = 0.65$

N	# of rays	# of cones	CPU (s)
5	8	1	0.11
7	24	3	0.31
9	47	5	0.87
11	100	9	2.78
13	273	15	10.89
15	744	30	74.05
17	2048	57	572.57
19	5413	108	3245.94



Ex.) plane truss

- $N = 5$ contact candidate nodes, $\mu = 1.0 \quad \rightarrow$ 25 extremal rays

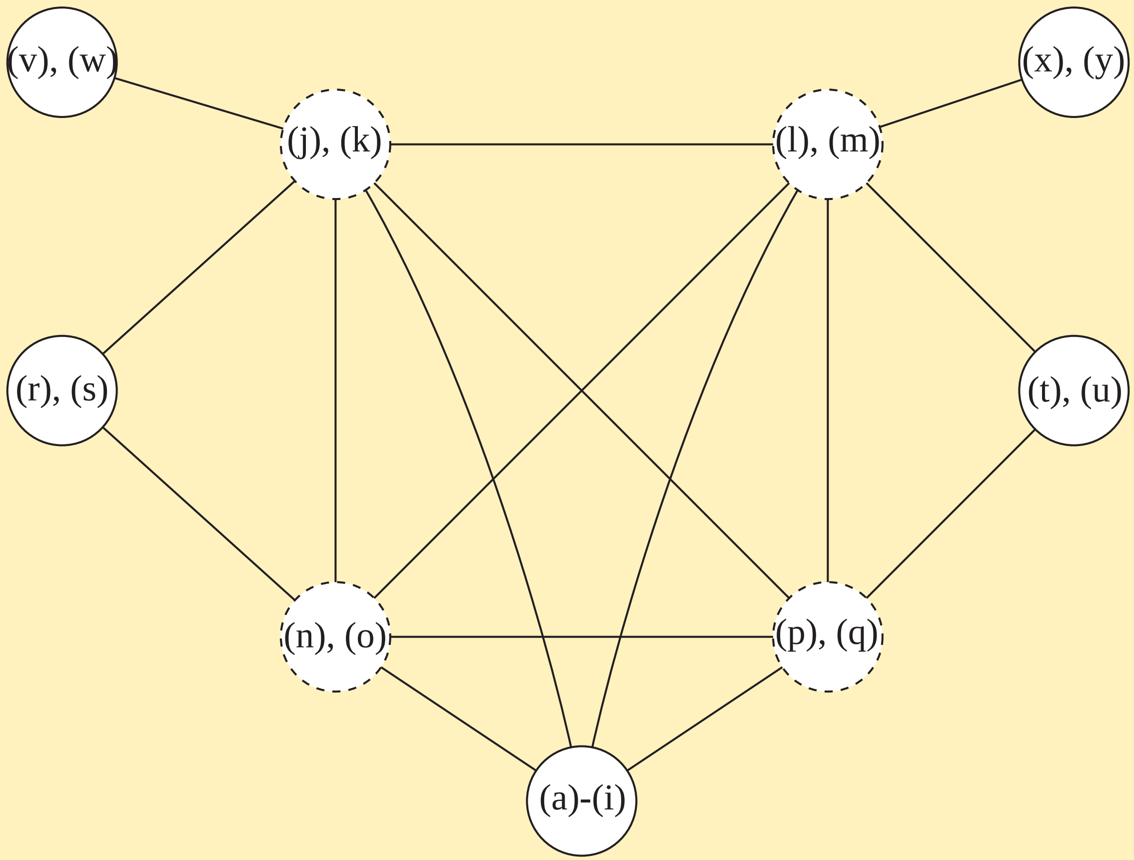


cross-complementarity relations

○ : strict complementarity

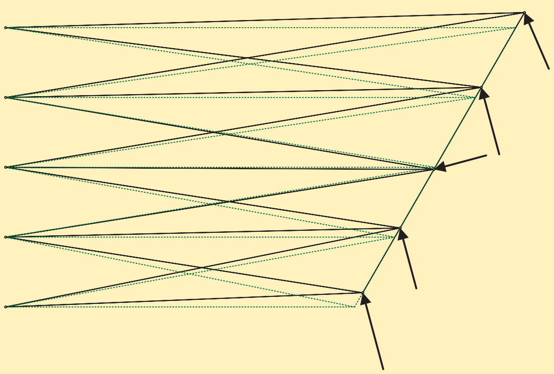
Ex.) plane truss

- $N = 5$ contact candidate nodes, $\mu = 1.0$ $\rightarrow$ 25 extremal rays

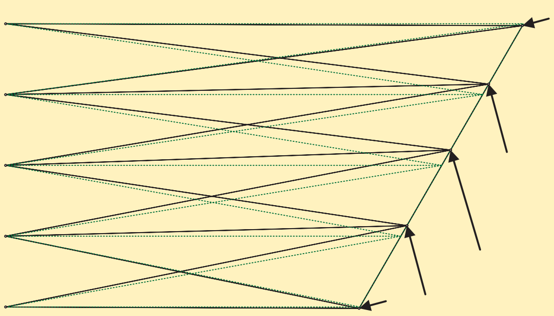


cross-complementarity relations

○ : strict complementarity



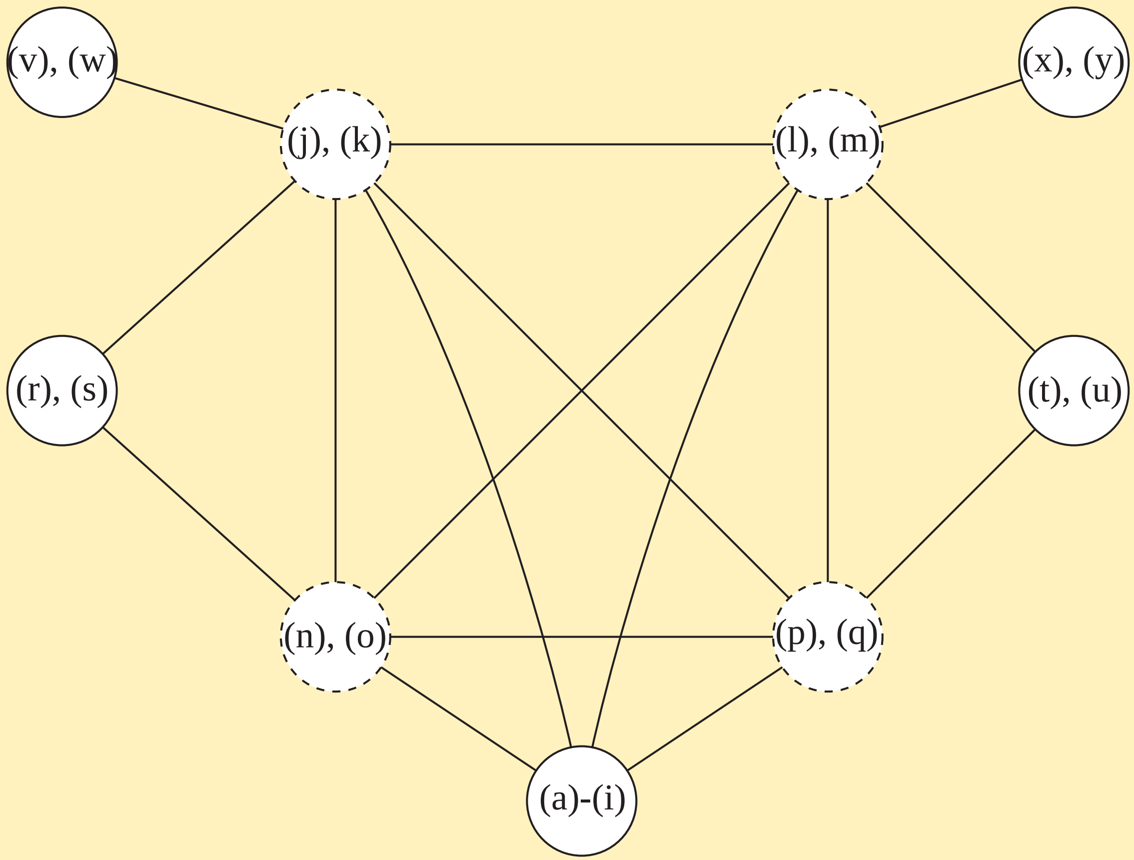
ex. ray (a)



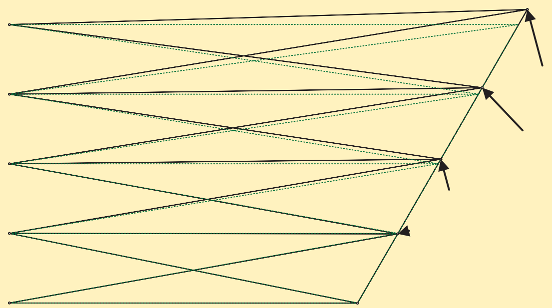
ex. ray (e)

Ex.) plane truss

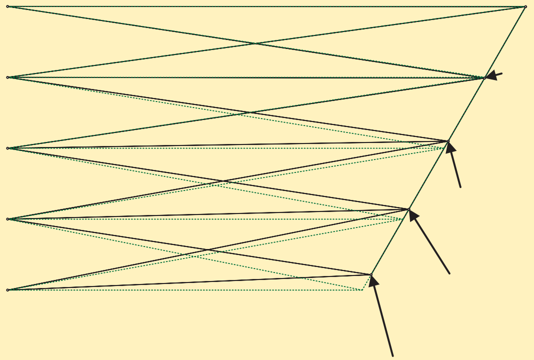
- $N = 5$ contact candidate nodes, $\mu = 1.0$ $\rightarrow$ 25 extremal rays



cross-complementarity relations
○ : strict complementarity



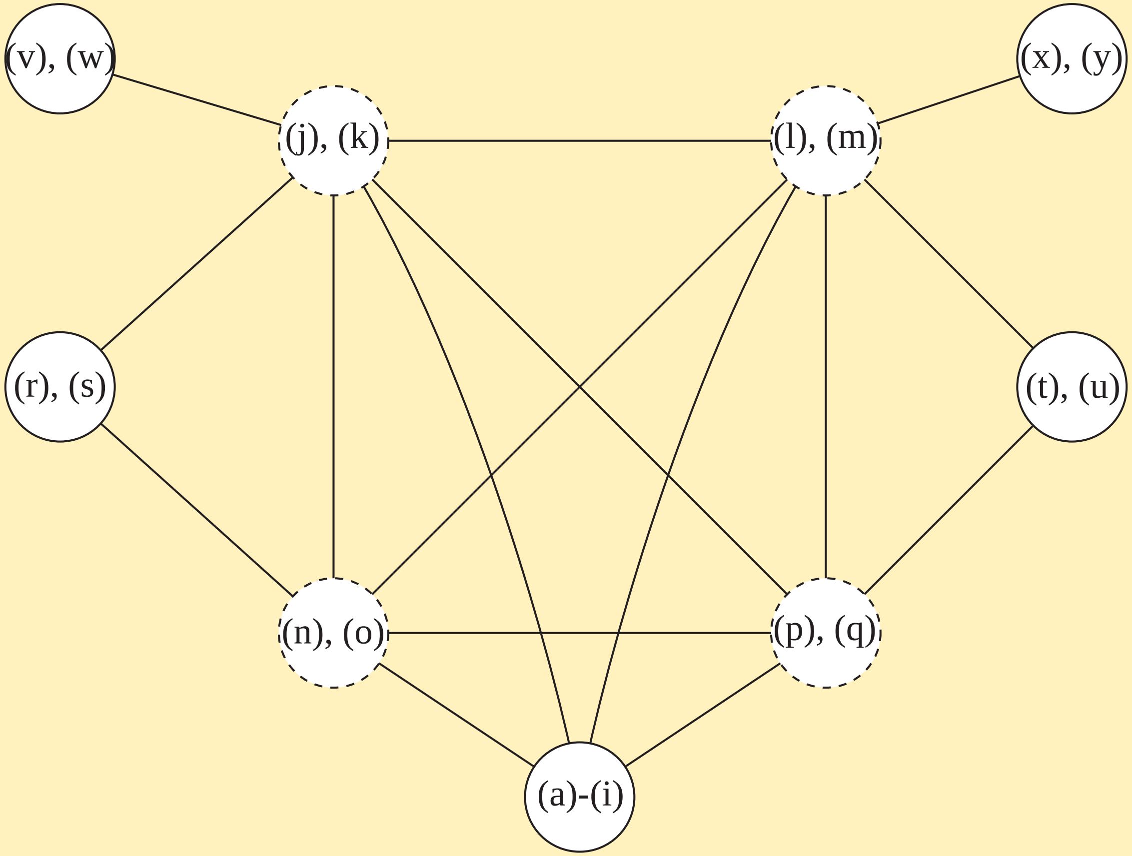
ex. ray (r)



ex. ray (t)

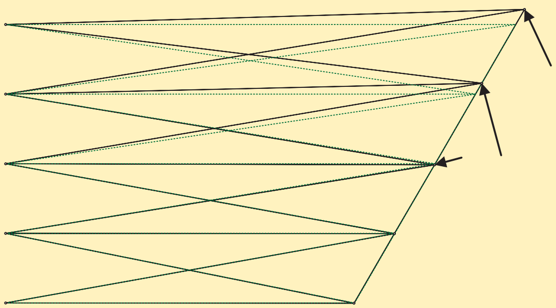
Ex.) plane truss

- $N = 5$ contact candidate nodes, $\mu = 1.0$ $\rightarrow$ 25 extremal rays

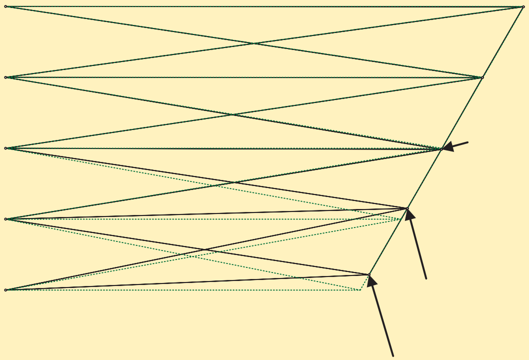


cross-complementarity relations

○ : strict complementarity



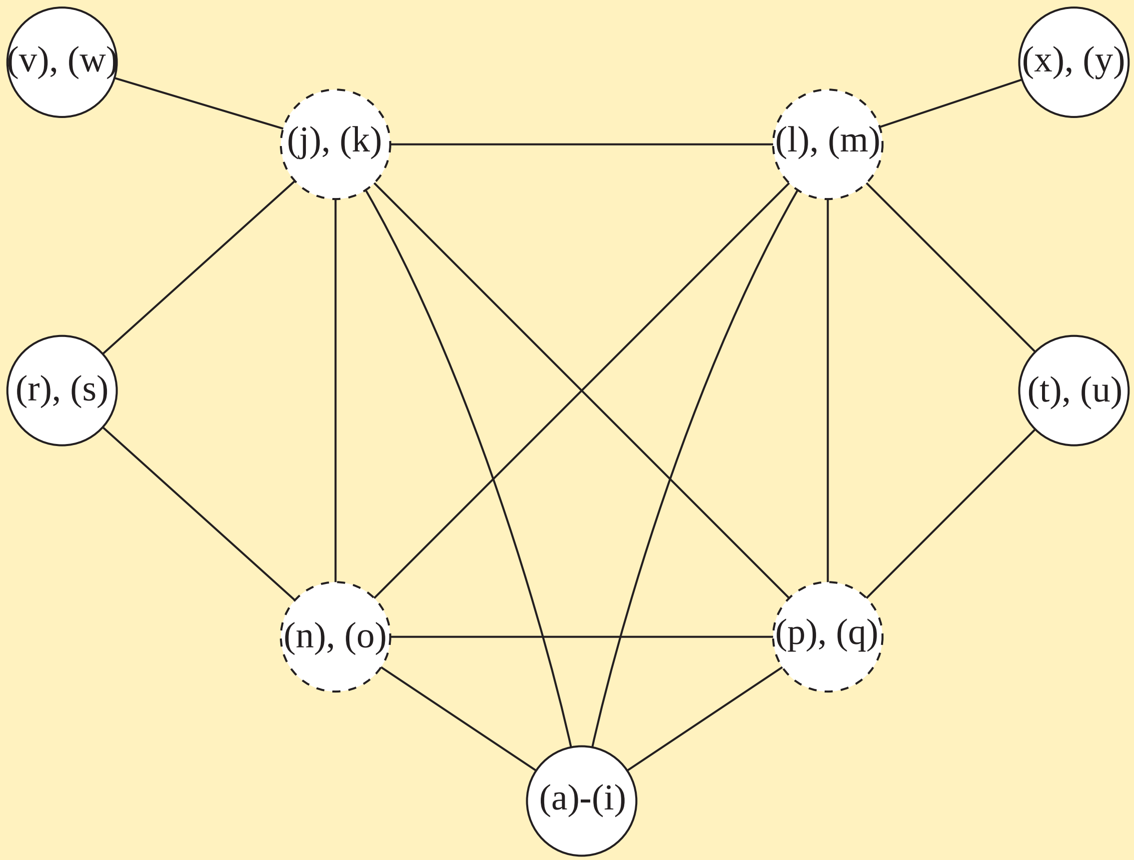
ex. ray (v)



ex. ray (x)

Ex.) plane truss

- $N = 5$ contact candidate nodes, $\mu = 1.0$ $\rightarrow$ 25 extremal rays



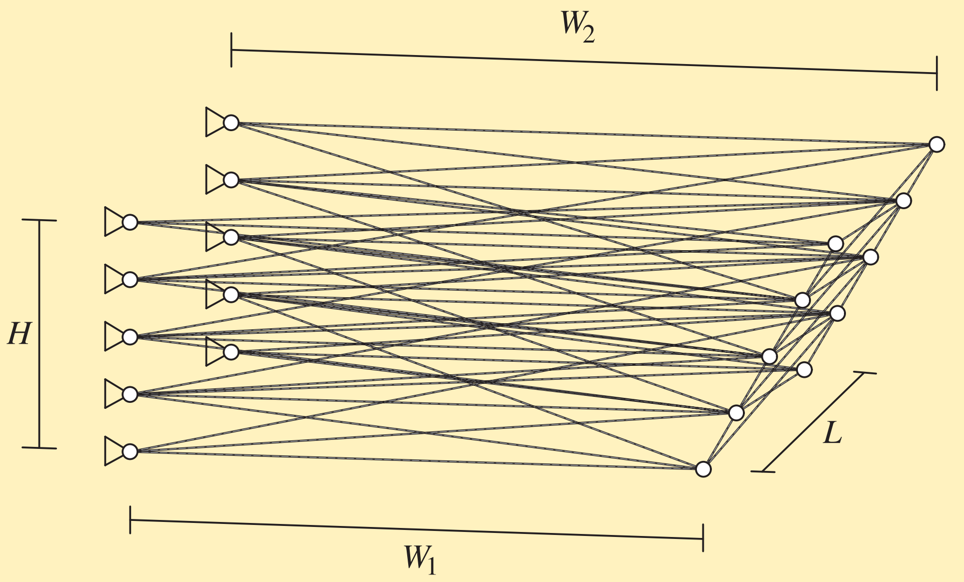
cross-complementarity relations

○ : strict complementarity

sign-pattern	μ^{c}
(a)–(i)	0.6072
(r), (s)	0.7804
(v), (w)	0.9422
(t), (u)	0.6718
(x), (y)	0.7909

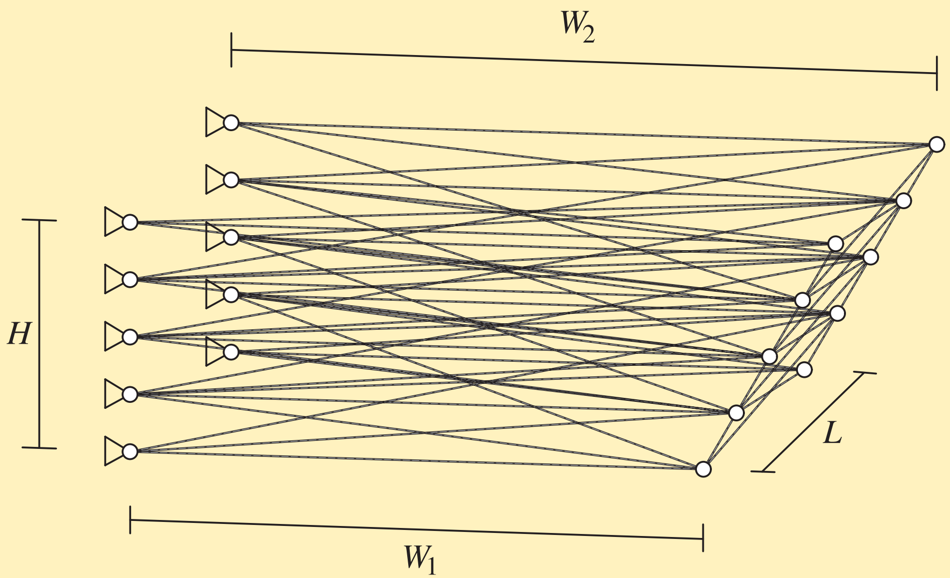
Ex.) 3D truss

- 10 contact candidate nodes
- branch-and-bound method
  for enumerating sign-patterns of $\boldsymbol{u}_{\rm n}$ & $\boldsymbol{r}_{\rm n}$



Ex.) 3D truss

- 10 contact candidate nodes
- branch-and-bound method
  for enumerating sign-patterns of $\boldsymbol{u}_{\rm n}$ & $\boldsymbol{r}_{\rm n}$
  - SOCP (second-order cone program) is solved
  - by interior-point method (SeDuMi 1.1 [Sturm 99], [Pólik 05])



μ	# of sign-patterns
0.55	0
0.60	1
0.65	26
0.70	61
1.00	236

conclusions

- wedging problem
  - finding nontrivial equilibrium configurations
  - unilateral contact with Coulomb friction
- solution set of 2D problem
  - enumeration of extremal rays
  - enumeration of cross-complementarity relation
- solution set of 3D problem
  - enumeration of sign-patterns of $\boldsymbol{u}_{\mathrm{n}}$ & $\boldsymbol{r}_{\mathrm{n}}$
- critical value of friction coefficient μ^{c}
  - global optimization
    based on the bisection method