Large Deflection Analysis of Cable Networks by Second-Order Cone Program

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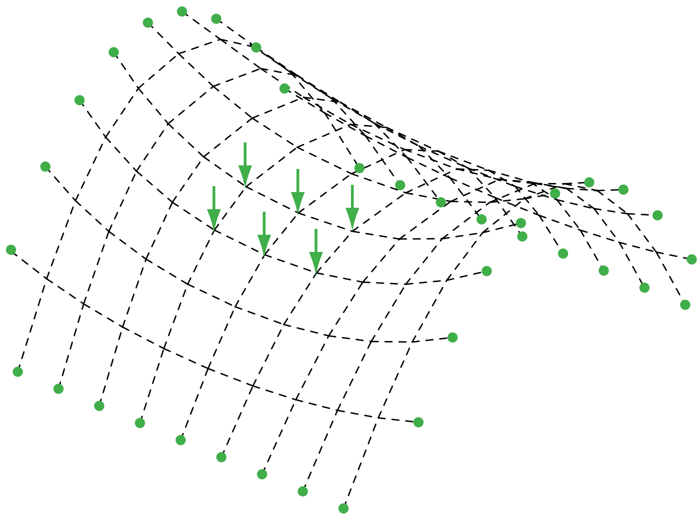
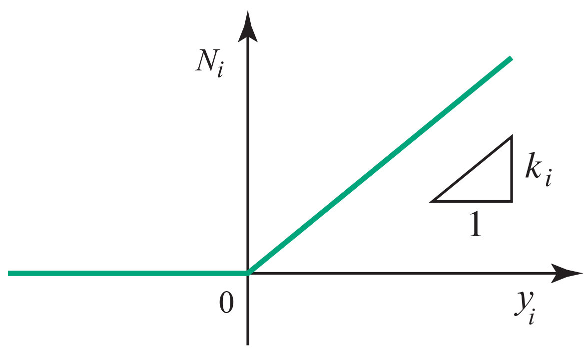
Large deformation analysis of cable network:

- Given
  - supports : $\boldsymbol{b}_i^0 \in \Re^3$
  - external forces : $\bar{\boldsymbol{f}} \in \Re^{3N^n}$
  - member unstressed length : l_i^0
- Find
  - internal nodes : $\boldsymbol{x} \in \Re^{3N^n}$



cable network



axial force–elongation

Stress unilateral behavior:

Backgrounds:

- transmit only tension force

  1. trial-and-error
     (a) assume whether in tensile or slackening state
     (b) check the obtained solution
     (c) correct the assumption
  2. singularity of the tangent stiffness matrix

The proposed algorithm:

1. Second-Order Cone Programming (SOCP) problem
   $\Longleftrightarrow$ min. of Total Potential Energy
2. pin-joints, frictionless joints
3. no assumption on stress state
4. convergence
   (a) unstable cable networks
   (b) estimate of stress states is difficult

Second-Order Cone Programming : SOCP

1. convex programming
2. including LP, QP, etc.
3. included in SDP
4. primal-dual interior-point method
   (Monteiro and Tsuchiya, 2000)
   - polynomial time convergence
5. applications
   (a) truss topology optimization (Jarre *et al.*, 1998)
   (b) magnetic shield design
   (Sasakawa and Tsuchiya, 2000)
   (c) antenna array design (Scholnik and Coleman, 2000)

Second-Order Cone Programming : SOCP

Minimize	$\boldsymbol{b}^\top \boldsymbol{y}$
subject to	$\boldsymbol{A}^\top \boldsymbol{y} + \boldsymbol{x} = \boldsymbol{c},$
	$x_0 \geq \lVert \boldsymbol{x}_1 \rVert.$

Variable vectors

$$\boldsymbol{x} = (x_0, \boldsymbol{x}_1) \in \Re^n, \quad \boldsymbol{y} \in \Re^m$$

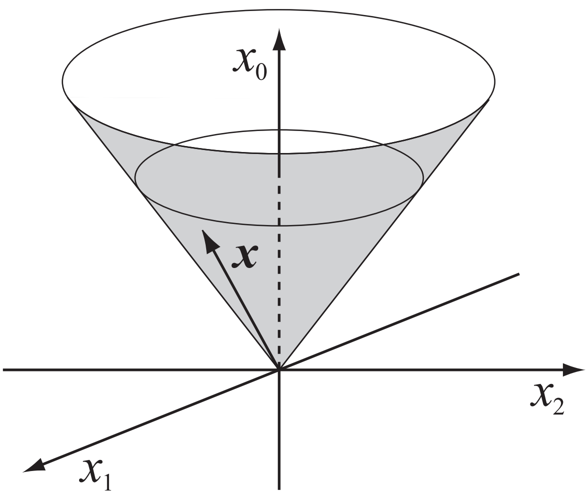
Definitions

$$\lVert \boldsymbol{x}_1 \rVert = (\boldsymbol{x}_1^\top \boldsymbol{x}_1)^{1/2}$$: Euclidean norm

$$\mathcal{K}(n) = \{(x_0, \boldsymbol{x}_1) | x_0 \geq \lVert \boldsymbol{x}_1 \rVert\}$$: second-order cone

Constant matrix and vectors

$$\boldsymbol{A} \in \Re^{m \times n}, \quad \boldsymbol{b} \in \Re^m, \quad \boldsymbol{c} \in \Re^n$$



Second-order cone in 3-dimensional space

Minimization problem of TPE

$$\mathrm{TPE}: \ \text{Minimize} \quad \sum_{i=1}^{N^m} w_i(y_i) - \bar{\boldsymbol{f}}^\top \boldsymbol{x}$$

$$\text{subject to} \quad w_i(y_i) = \begin{cases} \frac{1}{2}k_i y_i^2, & (y_i \geq 0), \\ 0, & (y_i < 0), \end{cases}$$

$$y_i = \|\boldsymbol{B}_i \boldsymbol{x} + \boldsymbol{b}_i^0\| - l_i^0.$$

Variables:

y_i : member elongation, $\boldsymbol{x}$: internal nodes

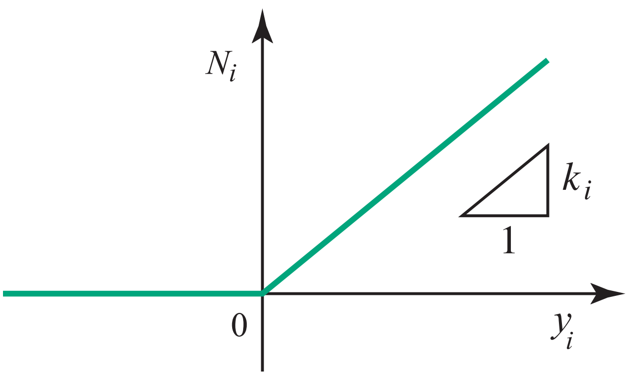
Given:

$\bar{\boldsymbol{f}}$: external forces, l_i^0 : member unstressed length,

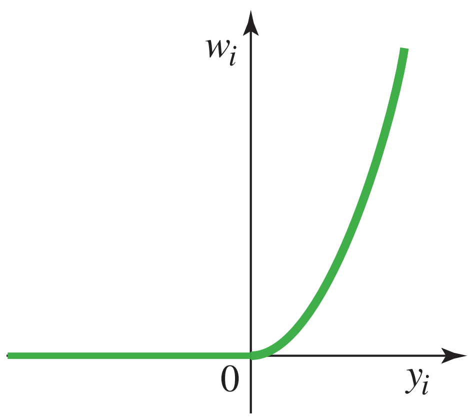
$\boldsymbol{b}_i^0$: supports, $\boldsymbol{B}_i^0$: adjacency matrices

Strain energy:

$w_i(y_i)$: strain energy. k_i : extensional stiffness



axial force–elongation



strain energy

1. nonconvex problem

2. w_i depends on the sign of y_i

   - assumptions are required

   - trial-and-error process

Minimization problem of TPE:

$$\text{TPE}: \quad \text{Minimize} \quad \sum_{i=1}^{N^m} w_i(y_i) - \bar{\boldsymbol{f}}^\top \boldsymbol{x}$$

$$\text{subject to} \quad w_i(y_i) = \begin{cases} \frac{1}{2} k_i y_i^2, & (y_i \geq 0), \\ 0, & (y_i < 0), \end{cases}$$

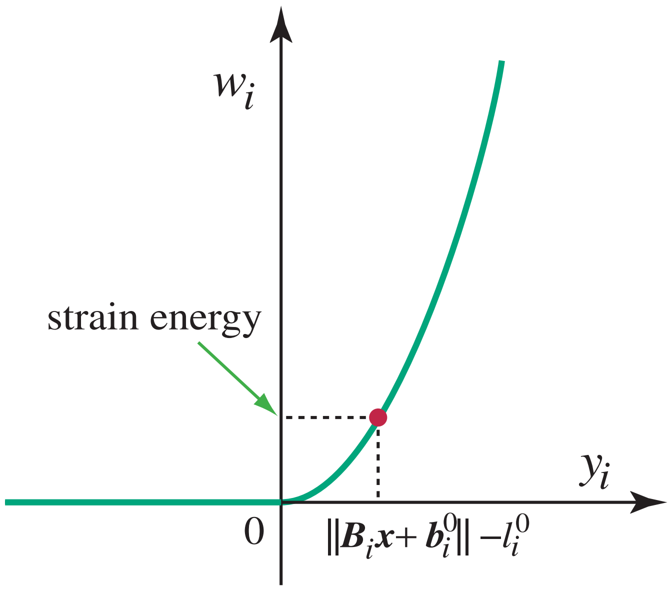
$$y_i = \|\boldsymbol{B}_i \boldsymbol{x} + \boldsymbol{b}_i^0\| - l_i^0.$$

SOCP formulation:

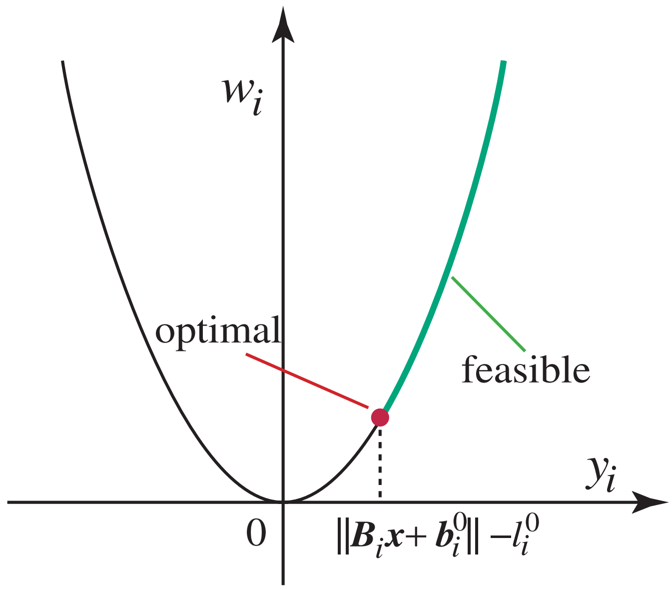
$$\text{SOCP}: \quad \text{Minimize} \quad \sum_{i=1}^{N^m} \frac{1}{2} k_i y_i^2 - \bar{\boldsymbol{f}}^\top \boldsymbol{x}$$

$$\text{subject to} \quad y_i \geq \|\boldsymbol{B}_i \boldsymbol{x} + \boldsymbol{b}_i^0\| - l_i^0.$$

in tensile state :

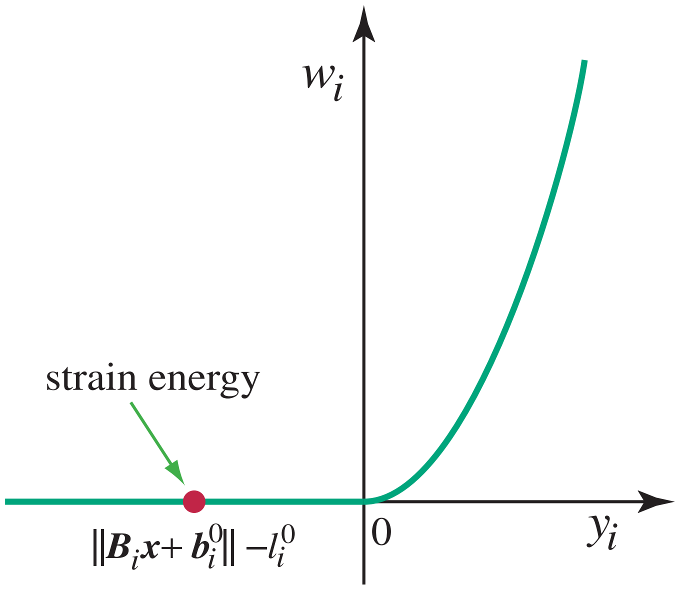


TPE

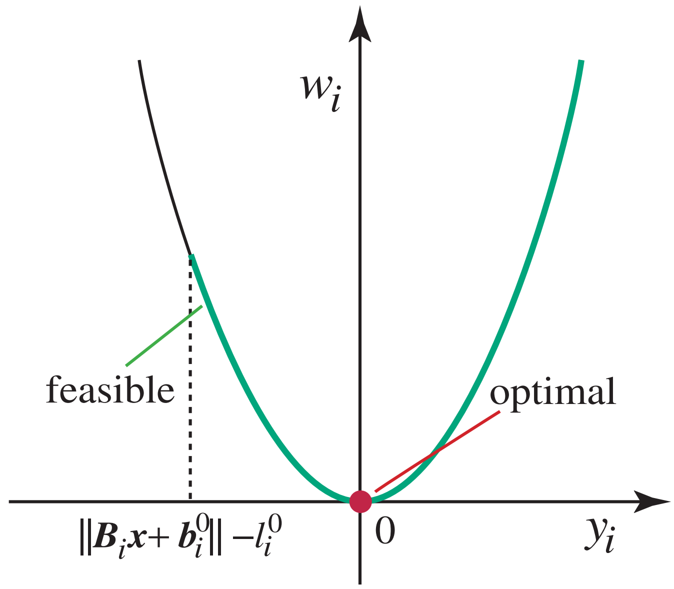


SOCP

slackening :



TPE

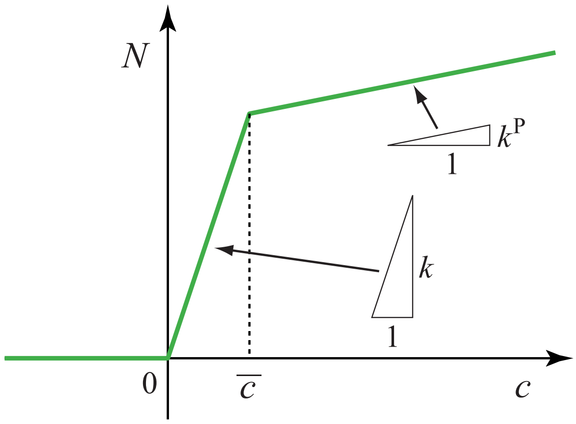


SOCP

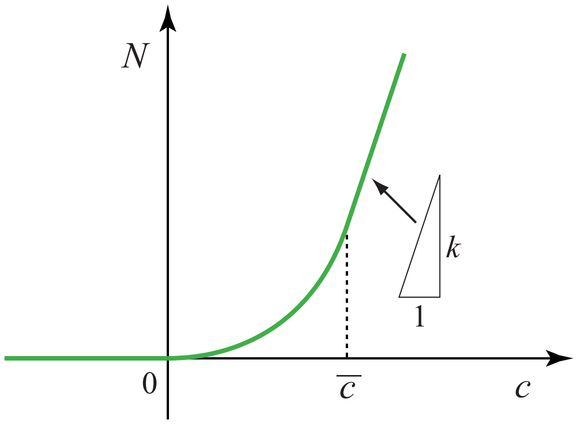
SOCP:

1. has the same optimizer as that of TPE
2. convex problem
3. efficient algorithm
   - no assumption is required—no trial-and-error process
   - polynomial-time convergence—IPM

Nonlinear constitutive law $\Longrightarrow$ SOCP:



Bi-linear material.



quadratic-linear constitutive law.

stiffness reduction:

1. small elongation
2. strand cable becomes lose
3. deflection of cable $\Longleftarrow$ own weight

Optimality conditions:

$$\text{SOCP}: \quad \text{Minimize} \quad \sum_{i=1}^{N^m} \frac{1}{2} k_i y_i^2 - \bar{\boldsymbol{f}}^\top \boldsymbol{x}$$
$$\text{subject to} \quad y_i \geq \|\boldsymbol{B}_i \boldsymbol{x} + \boldsymbol{b}_i^0\| - l_i^0.$$

Lagrangian multipliers

$$q_i \in \Re, \quad \boldsymbol{v}_i \in \Re^3, \quad (i = 1, 2, \cdots, N^m).$$

KKT conditions:

$$q_i = k_i y_i, \qquad : \text{constitutive law}$$
$$\sum_{i=1}^{N^m} \boldsymbol{B}_i^\top \boldsymbol{v}_i + \bar{\boldsymbol{f}} = \boldsymbol{0}, \qquad : \text{equilibrium equations}$$
$$(y_i + \bar{l}_i^0) q_i + (\boldsymbol{B}_i \boldsymbol{x} + \boldsymbol{b}_i^0)^\top \boldsymbol{v}_i = 0,$$
$$y_i \geq \|\boldsymbol{B}_i \boldsymbol{x} + \boldsymbol{b}_i^0\| - \bar{l}_i^0, \quad q_i \geq \|\boldsymbol{v}_i\|.$$

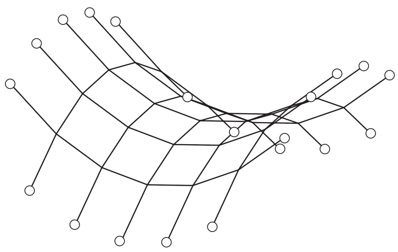
- q_i : axial force
- $\boldsymbol{v}_i$: internal force vector

Examples:

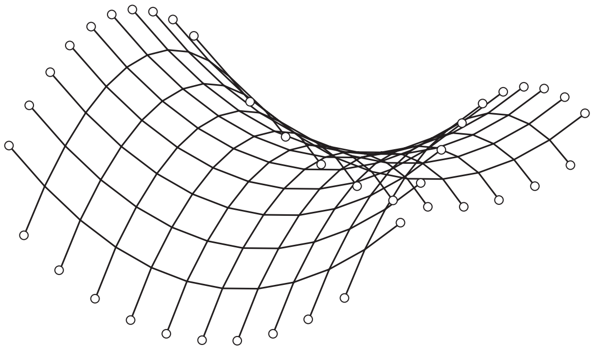
SOCP : solve SOCP problem by IPM.

TPE : solve min. of TPE by IPM.

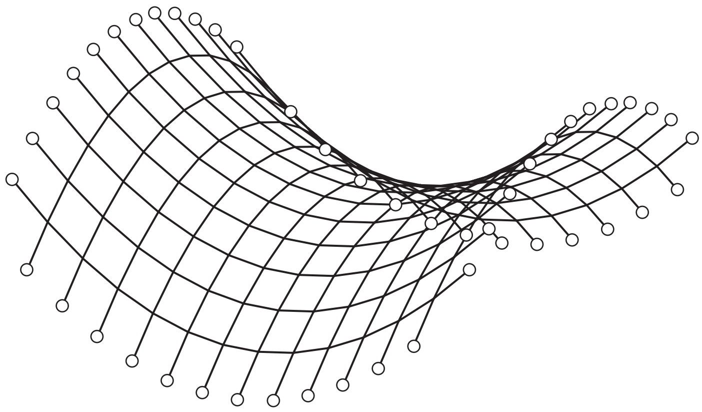
NR : Newton-Raphson method (tangent stiffness).



Model (I)



Model (II)

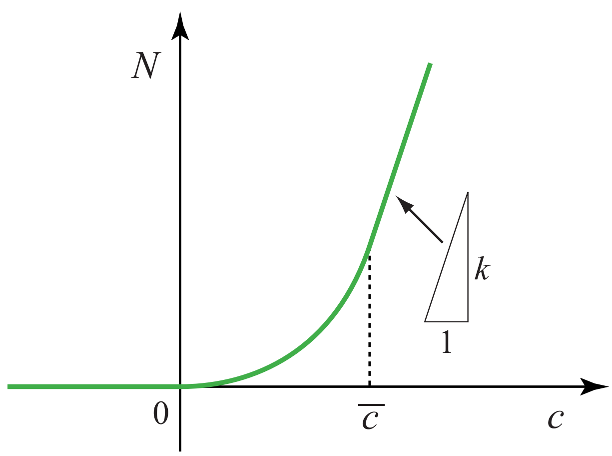


Model (III)

Self-equilibrium shape analysis

algorithm	model #	model n	steps	CPU time (sec.) total	CPU time (sec.) ave.
SOCP	(I)	195	11	0.23	0.021
	(II)	740	14	1.42	0.101
	(III)	1056	15	2.46	0.164
TPE	(I)	135	22	0.50	0.023
	(II)	520	22	2.44	0.111
	(III)	744	22	3.81	0.173
NR	(I)	75	19	0.28	0.014
	(II)	300	22	4.71	0.214
	(III)	432	22	20.31	0.923

1. CPU time
   - $\mathcal{O}(n)$ < SOCP < TPE < $\mathcal{O}(n^2)$
   - $\mathcal{O}(n^2)$ < NR < $\mathcal{O}(n^3)$
2. # of variables : n
   - SOCP > TPE > NR



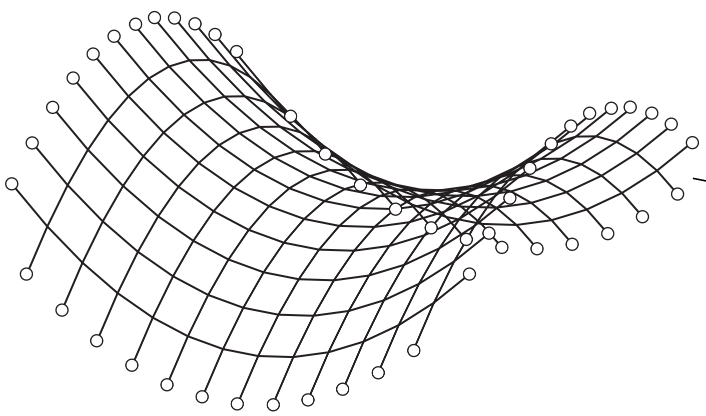
Dependence of initial solutions

algorithm	model #	n	(A) steps	(A) CPU (sec.)	(B) steps	(B) CPU (sec.)
SOCP	(I)	135	11	0.21	15	0.22
	(II)	520	11	0.90	15	0.98
	(III)	744	11	1.42	23	2.26
TPE	(I)	135	11	0.24	15	0.23
	(II)	520	11	0.93	15	1.05
	(III)	744	15	2.10	31	3.28
NR	(I)	75	22	0.26	*	*
	(II)	300	20	4.24	*	*
	(III)	432	9	7.40	*	*

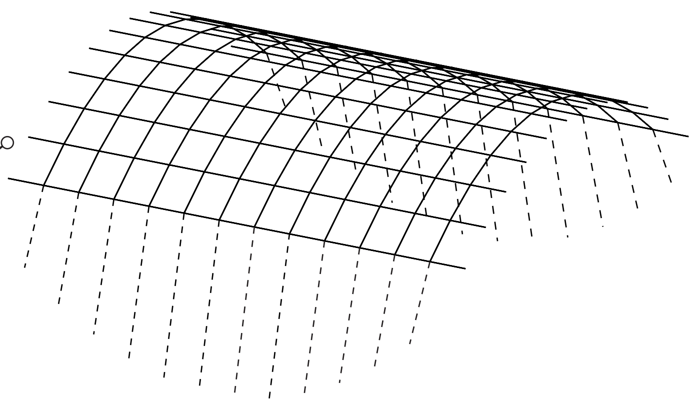
1. SOCP (A) ≒ SOCP (B)

2. NR (B) : fail

   tangent stiffness matrix : singular



Initial solution (A).



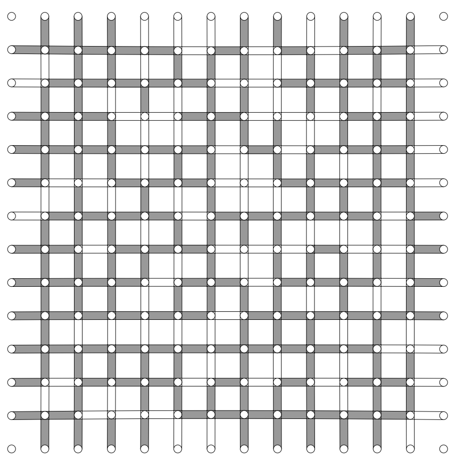
Initial solution (B).

Initial configuration with many slackening members

algorithm	model		steps	CPU time (sec.)	
	constitutive law	n		total	ave.
SOCP	linear	744	45	5.37	0.120
SOCP	quadratic	1056	34	5.26	0.155
TPE	linear	744	51	6.37	0.125
TPE	quadratic	744	**	**	0.978
NR	linear	432	**	**	0.873
NR	quadratic	432	**	**	1.122

** : did not converge within 1000 steps.

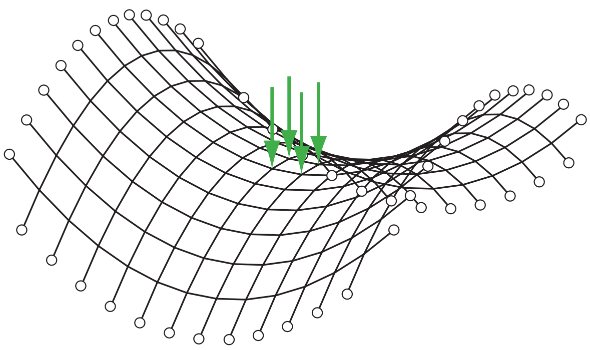
1. SOCP : converge
2. TPE, NR : fail (oscillation)
    - trial-and-error process

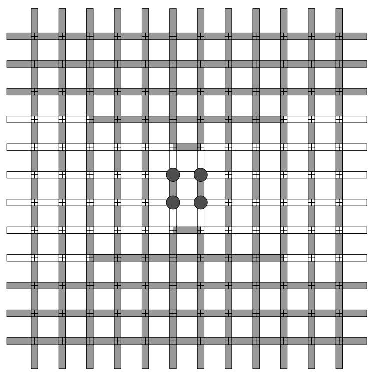


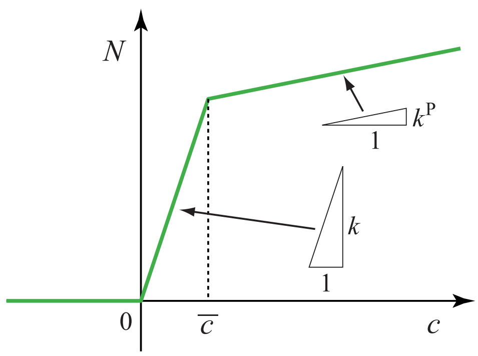
Model (III'): Slackening members.

Bi-linear material

algorithm	model		steps	CPU time (sec.)	
	#	n		total	ave.
SOCP	(I)	195	10	0.44	0.044
	(II)	740	10	1.39	0.139
	(III)	1056	10	2.06	0.206
TPE	(I)	135	10	0.58	0.058
	(II)	520	11	1.62	0.147
	(III)	774	11	2.17	0.197
NR	(I)	75	25	0.32	0.013
	(II)	300	32	6.92	0.194
	(III)	432	34	31.66	0.931


External loads.


Members in plastic state



Conclusions:

1. An SOCP formulation has been proposed for large deformation analysis of cable networks.
   (a) pin-joints
   (b) frictionless joints
   (c) nonlinear material
2. Equilibrium configurations are obtained by solving SOCP problems by using Interior Point Method.
   (a) no assumption on stress state
   (b) no process of trial and error
3. SOCP formulation is more efficient than
   (a) min. of Total Potential Energy.
   (b) Newton-Raphson method based on tangent stiffness.