Truss Topology Optimization under Constraints on Number of Different Design Variables

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constraint on “# of different design variables”

- a new modeling of design constraints
  - optimal standardization / optimal grouping
- a global optimization approach
  - MISOCP (mixed-integer second-order cone programming)

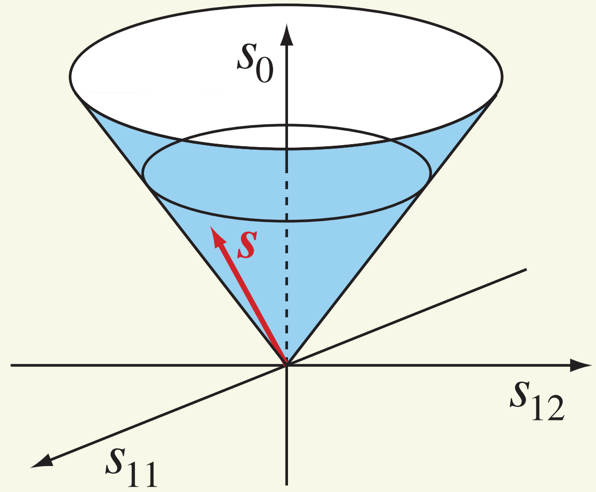
second-order cone programming

- SOCP:

$$\begin{array}{ll} \text{Min.} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s. t.} & \boldsymbol{p}_i^\top \boldsymbol{x} + q_i \geq \|\boldsymbol{c}_i - A_i \boldsymbol{x}\| \quad (i = 1, \dots, m) \end{array}$$

  - nonlinear convex optimization
  - primal-dual interior-point method

- SOC:



MISOCP (= mixed-integer second-order cone programming)

- "integrality" + SOC:

$$\begin{aligned} \text{Min.} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s. t.} \quad & \boldsymbol{p}_i^\top \boldsymbol{x} + q_i \geq \|\boldsymbol{c}_i - A_i \boldsymbol{x}\| \quad (i = 1, \dots, m) \\ & x_1, \dots, x_r \in \{0, 1\} \\ & x_{r+1}, \dots, x_n \in \mathbb{R} \end{aligned}$$

  - some discrete variables & some continuous variables

- relax $x_j \in \{0, 1\}$ as $0 \leq x_j \leq 1$ $\rightarrow$ SOCP
  - solvable with, e.g., a branch-and-bound method

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  - some discrete variables & some continuous variables

- relax $x_j \in \{0, 1\}$ as $0 \leq x_j \leq 1$ → SOCP
  - solvable with, e.g., a branch-and-bound method

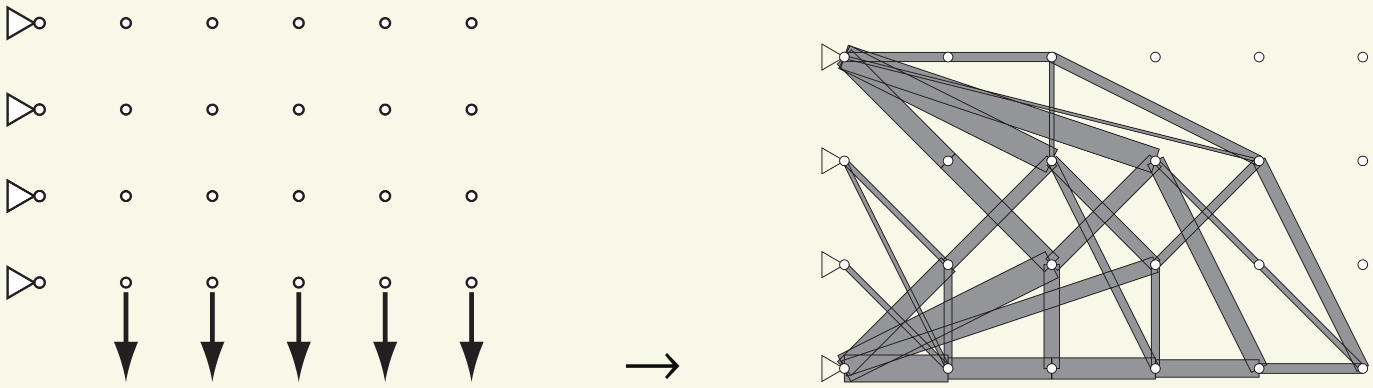
- higher modeling ability than MILP
- well-developed solvers
  - commercial: CPLEX, Gurobi Optimizer, XPRESS
  - non-commercial: SCIP [Achterberg '09]

m-i prog. in structural optimization

- m-i linear prog.:
  - continuum, absence (0) or presence (1) of elements [Stolpe & Svanberg '03]
  - truss, discrete c.-s. areas [Rasmussen & Stolpe '08] [Mela 14]
  - tensegrity (cable–strut structure) [K. '12, '13]
- m-i nonlinear prog.:
  - truss, discrete c.-s. areas [Achtziger & Stolpe '07] [Cerveira, Agra, Bastos & Gromicho '13] [Stolpe '14]
  - truss, continuous c.-s. areas [Ringertz '86] [Ohsaki & Katoh '05]
  - link mechanism [Stolpe & Kawamoto '05]
- m-i quadratic prog.:
  - truss, discrete c.-s. areas [Achtziger & Stolpe '09]

motivation: truss optimization

- conventional compliance optimization:



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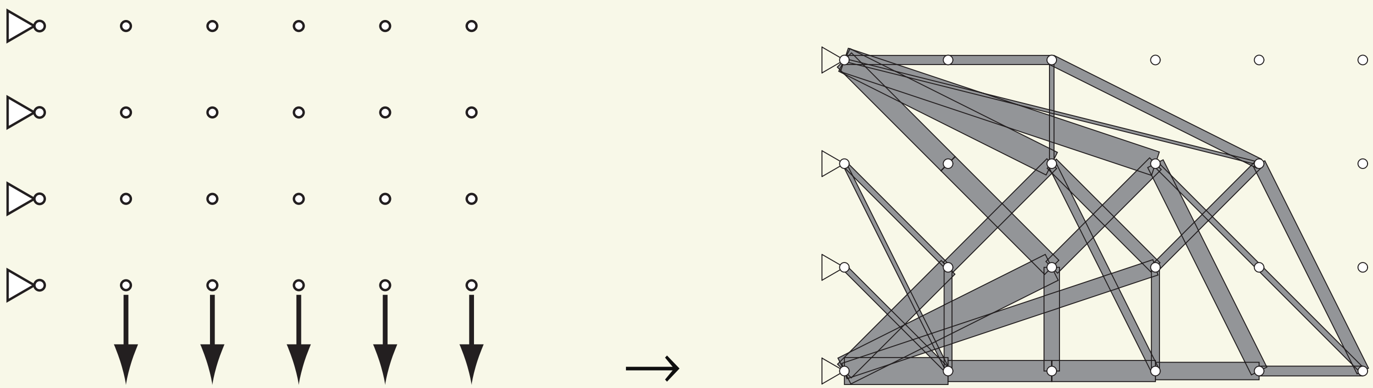

  - (almost) all members have different cross-sectional areas
    - 29 members have 25 different c-s areas
    - practically unrealistic

motivation: truss optimization

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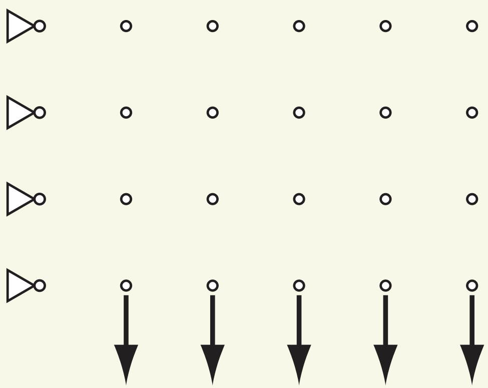
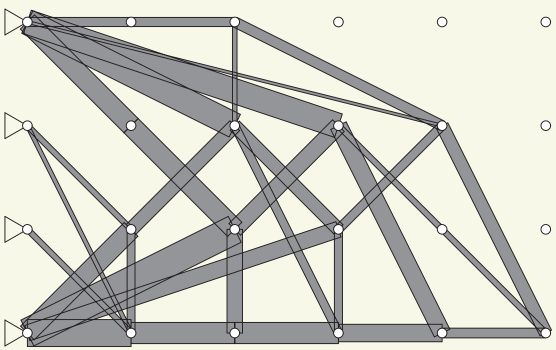
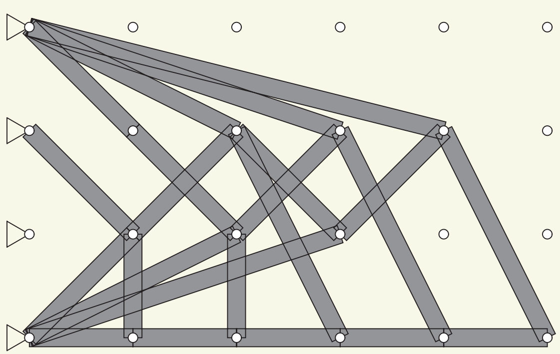


→



- opt. sol. w/ uniform c-s areas:

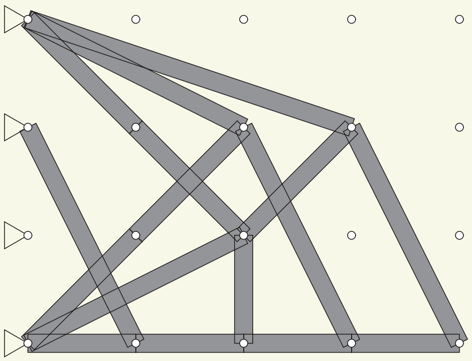


  - high manufacturability

- → new constraint: "# of different c-s areas" $\leq n$
  - uniform ⇔ $n = 1$

uniformity constraint ($n = 1$)

- two decisions:
  - Determine whether member e exists or vanishes. [combinatorial]
  - Determine the opt. val. for the common c-s area. [continuous]



$\rightarrow$ Both are addressed simultaneously.

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- two decisions:
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- $x_1, \dots, x_m$: member c-s areas
  - additional variables: $t_1, \dots, t_m \in \{0, 1\}$, $y \in \mathbb{R}$

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- $x_1, \dots, x_m$: member c-s areas
  - additional variables: $t_1, \dots, t_m \in \{0, 1\}$, $y \in \mathbb{R}$

$$x_e = \begin{cases} y & \text{if } t_e = 1 \qquad \text{[exists]} \\ 0 & \text{if } t_e = 0 \qquad \text{[vanishes]} \end{cases}$$

  - reformulation to linear inequalities:

$$0 \le x_e \le x^{\max} t_e$$
$$|x_e - y| \le x^{\max}(1 - t_e)$$

    - $y \in \mathbb{R}$ is a continuous design variable.
    - $x^{\max}$: a constant (upr. bd. for x_e)

generalization: up to n different values

- two decisions:
  - Determine optimal values for c-s areas, $y_1, \dots, y_n$.
  - Determine which x_e chooses out of $0, y_1, \dots, y_n$.

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- 0-1 variables
  - $s_{e1}, \ldots, s_{en} \in \{0, 1\}$: selection among $y_1, \ldots, y_n$

$$x_e = y_j \quad \Leftrightarrow \quad s_{ej} = 1$$

  - $t_e \in \{0, 1\}$: existence/absence of member e

$$t_e = s_{e1} + \cdots + s_{en}$$

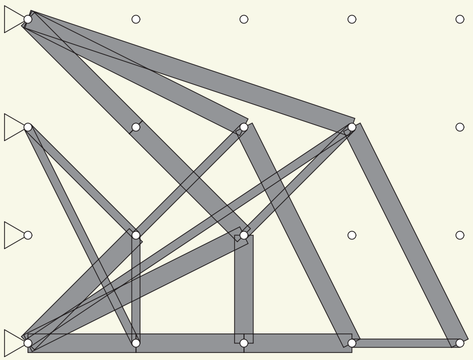
- reformulation to linear inequalities:

$$0 \le x_e \le x^{\max} t_e$$
$$|x_e - y_j| \le x^{\max}(1 - s_{ej})$$

  - $y_1, \ldots, y_n \in \mathbb{R}$ are continuous design variables

meaning of the constraints

- example: $n = 2$



opt. sol.

meaning of the constraints

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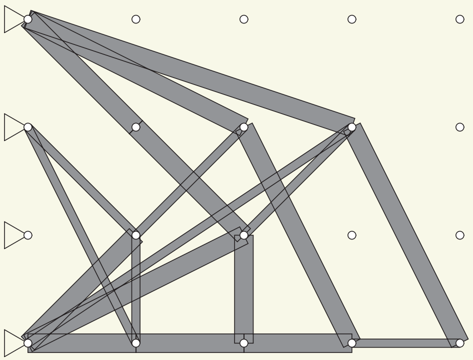

- Members are divided into...
  - set (A): Members whose c-s areas are 0.
  - set (B): Members whose c-s areas are y_1.
  - set (C): Members whose c-s areas are y_2.

- two factors:
  - To which set member e belongs? [optimal grouping]
  - What are the optimal values for y_1 & y_2? [optimal sizing]

for compliance optimization...

- conventional compliance optim.:

$$\begin{array}{ll} \text{Min.} & \text{(compliance)} \\ \text{s. t.} & \text{(vol. cstr.)}, \ \text{(c-s area} \geq 0) \end{array}$$

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- (compliance) $= 2\times$ (complementary energy)
  - cstr.: force-balance eq.

    $\rightarrow$ SOCP

[Ben-Tal & Nemiroviski ’01] [Jarre, Kočvara & Zowe ’98]

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- compliance optim. w/ cstr. on # of different c-s areas

  $\rightarrow$ MISOCP **(!)**
  - It can be solved globally.

basis: SOCP for compliance optimization

- SOCP formulation:

$$\begin{aligned} \text{Min.} \quad & 2(w_1 + \cdots + w_m) \\ \text{s. t.} \quad & w_e + x_e \geq \left\| \begin{bmatrix} w_e - x_e \\ \sqrt{2l_e/E}q_e \end{bmatrix} \right\| \qquad (\spadesuit) \\ & \sum_{e=1}^{m} q_e \boldsymbol{b}_e = \boldsymbol{f}, \quad \text{(vol. cstr.)} \end{aligned}$$

[Jarre, Kočvara & Zowe '98] [Ben-Tal & Nemiroviski '01]

  - variables:

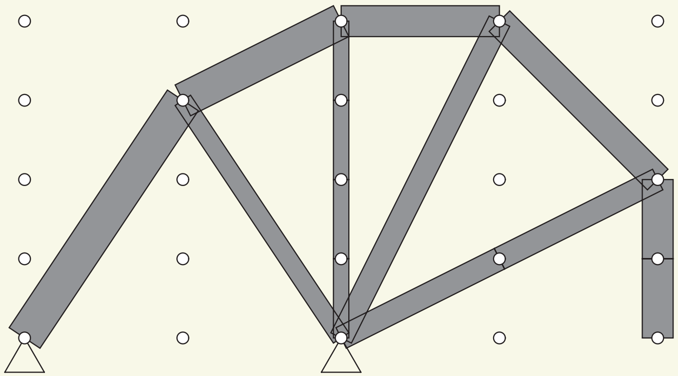
    x_e : c-s area, q_e : axial force, w_e : complementary strain energy

- If $x_e > 0$,

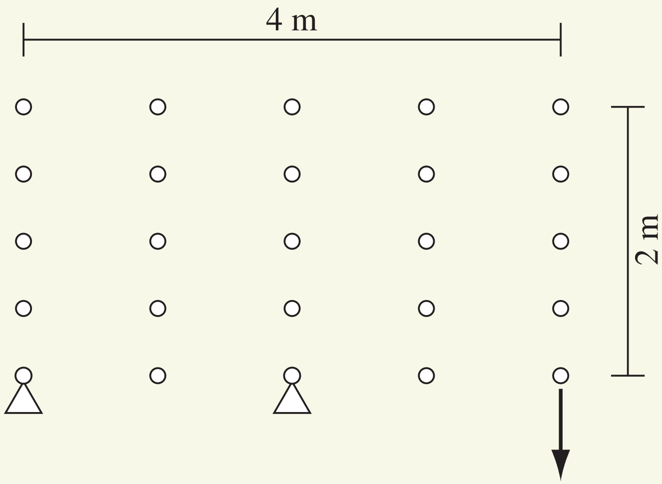
$$(\spadesuit) \quad \Leftrightarrow \quad w_e \geq \frac{1}{2}\frac{l_e}{Ex_e}q_e^2$$

i.e., def. of complementary strain energy.

ex.） 200-bar truss

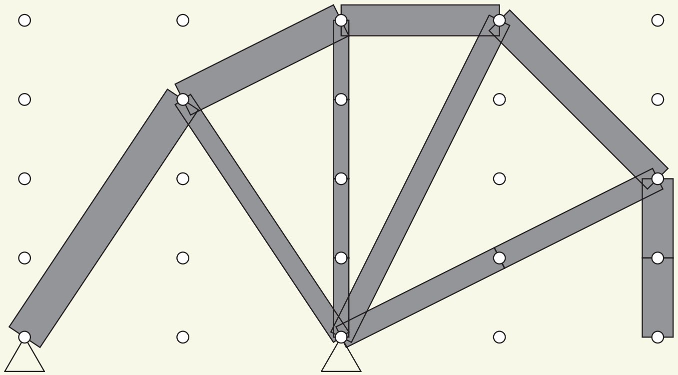


continuous optim.



ground structure

ex.）200-bar truss



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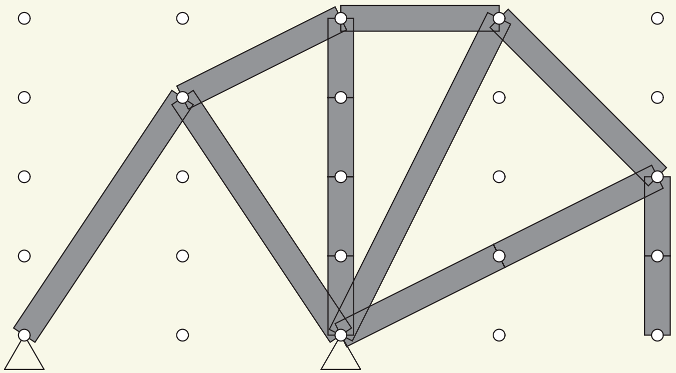

$n = 1$: uniform

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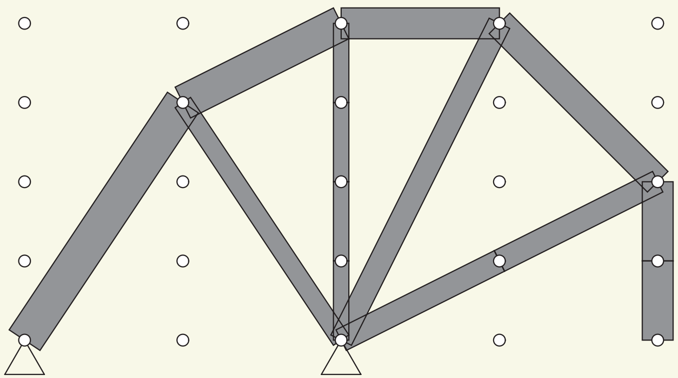

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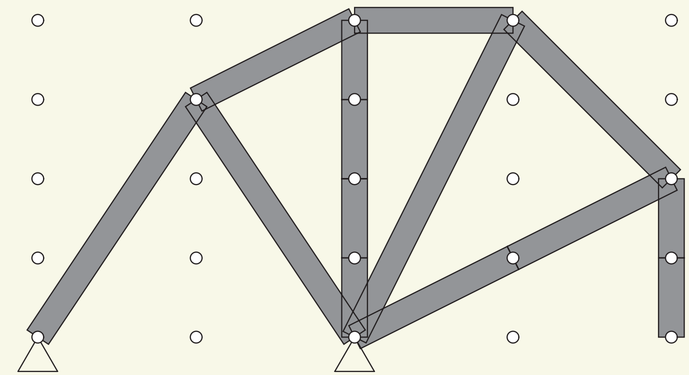

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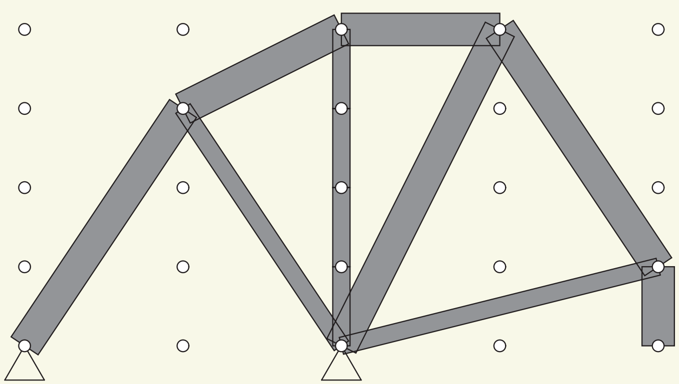
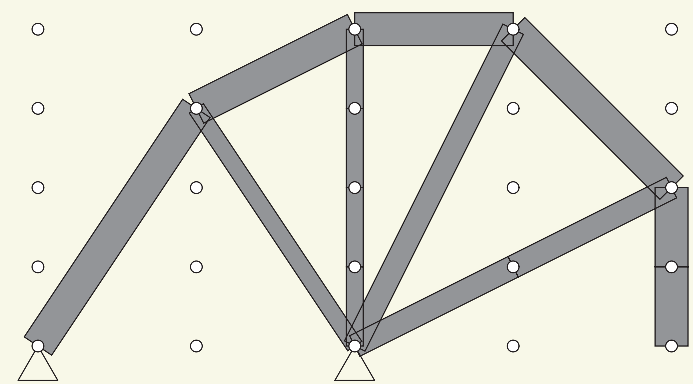
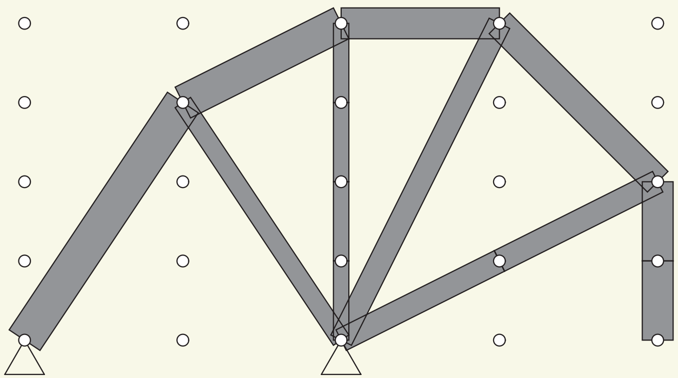


$n = 2$: two groups

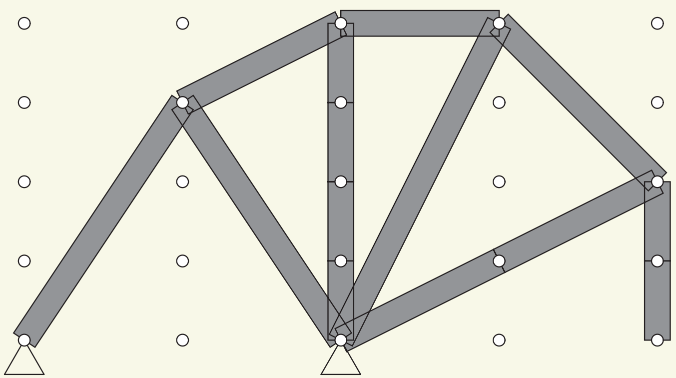


$n = 3$: three groups

ex.） 200-bar truss



continuous optim. 462.59 J

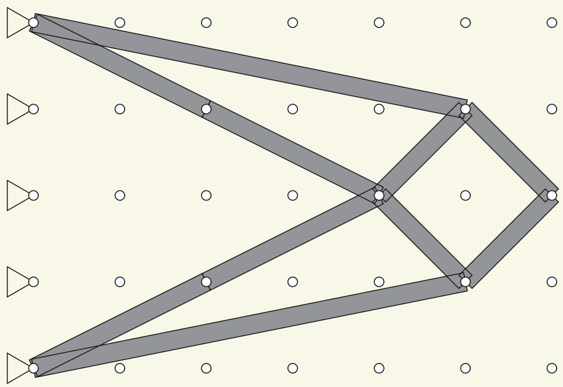


$n = 1$: uniform

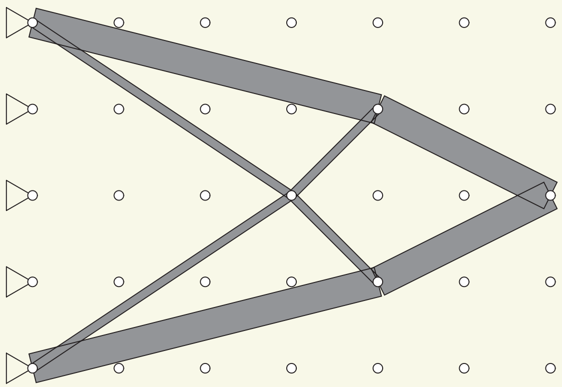
		Areas (mm^2)				
n	Obj. (J)	y_1	y_2	y_3	Time (s)	# of 0-1 vrbs.
1	497.38	1095.15	—	—	13.2	200
2	469.55	1380.90	736.34	—	16.2	400
3	465.63	1403.07	976.98	721.14	33.5	600

- Member standardization is realized with small increase of opt. val.

ex.) cantilever truss



$(N_X, N_Y) = (6, 4), n = 1$



$(N_X, N_Y) = (6, 4), n = 2$

(N_X, N_Y)	n	# of membs.	# of 0-1 vrbs.	Time (s)	
$(6, 4)$	1	386	386	202.1	
$(6, 4)$	2	386	772	137.7	
$(7, 4)$	1	503	503	278.2	
$(7, 4)$	2	503	1006	1,015.8	
$(5, 6)$	1	559	559	2,778.5	
$(6, 6)$	1	748	748	24,850.6	$(< 7\,\mathrm{h})$

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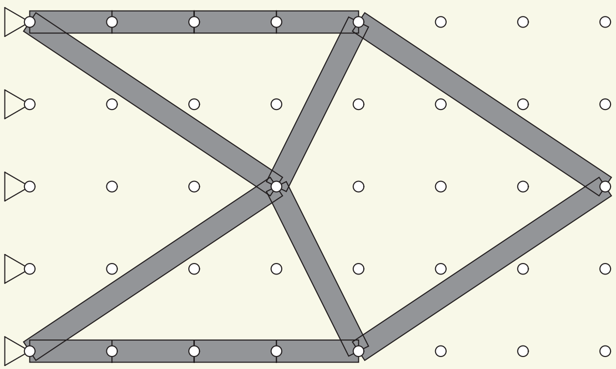

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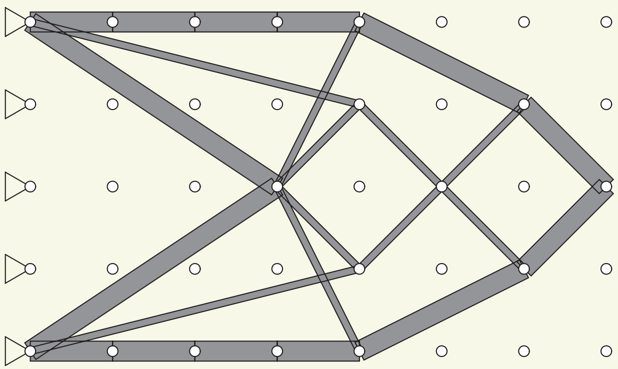

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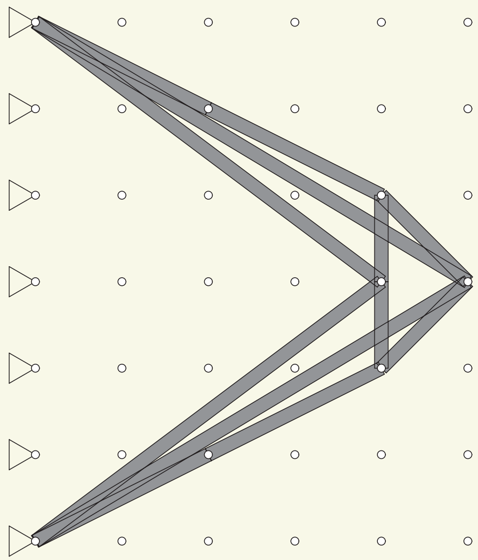
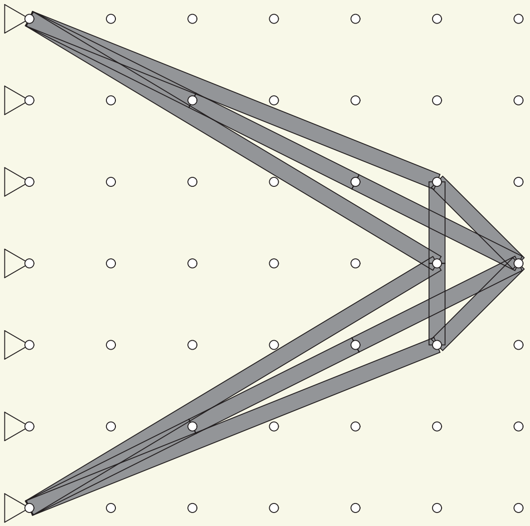


$(N_X, N_Y) = (5, 6)$, $n = 1$



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a by-product

- MISOCP is efficient for “discrete truss optimization”,...
  - ...compared with MILP/MIQP.
  - **(!)** MILP [Rasmussen & Stolpe ’08] & MIQP [Achtziger & Stolpe ’09] can handle, e.g., stress cstr.

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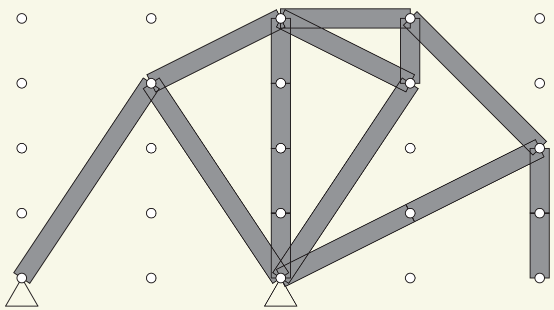
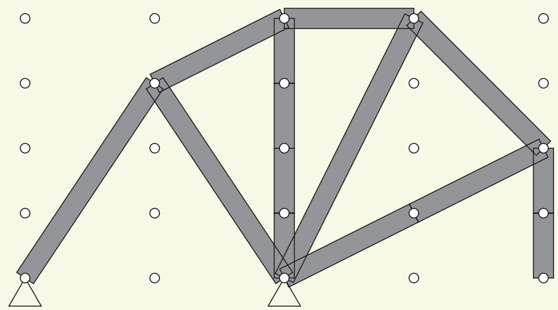
- comparison:
  - Choose c.-s. areas as $x_1, \dots, x_m \in \{0, \bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_r\}$,
    - w/o cstr. on # of different c-s areas
    - compliance optim.
  - $m = 200$, $r = 1, 2, 3$:
    - “MISOCP $< 20\,\mathrm{s}$” vs. “MILP, MIQP $> 24\,\mathrm{h}$”
  - MILP & MIQP use big-M.
  - Relaxation of MISOCP has nonlinear (convex) constraints.

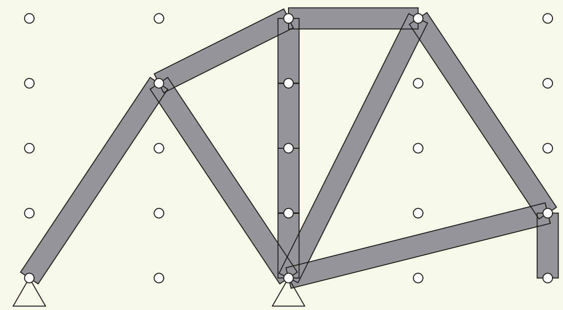
ex.) 200-bar truss: discrete optimization



$x_e \in \{0, 1000\}$



$x_e \in \{0, 1050\}$



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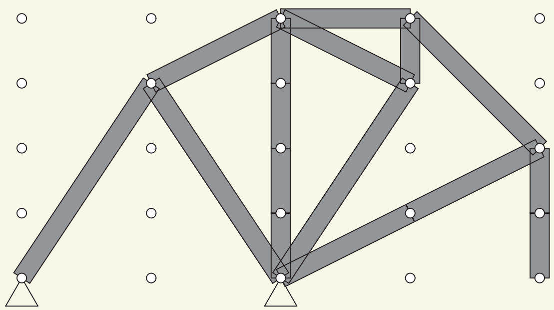

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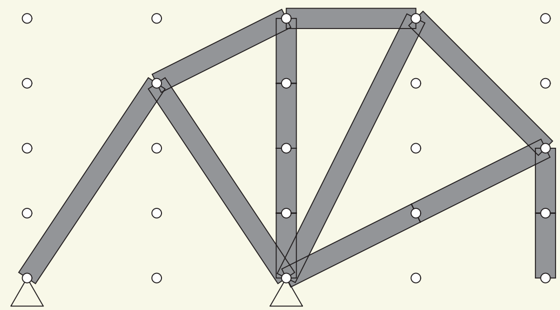

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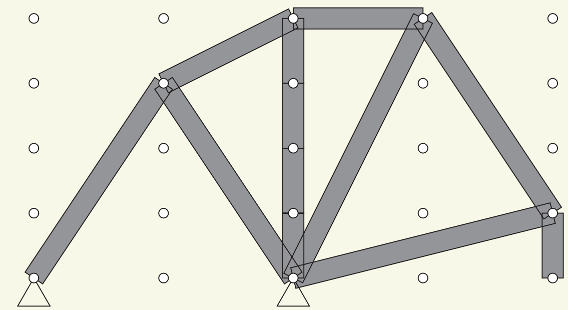
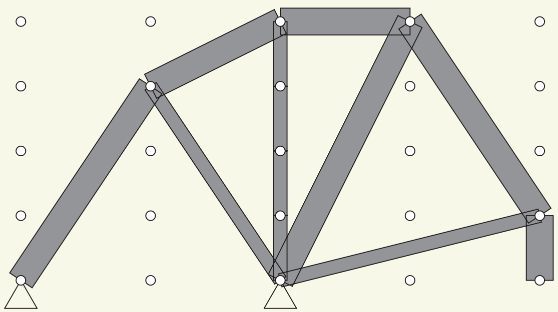
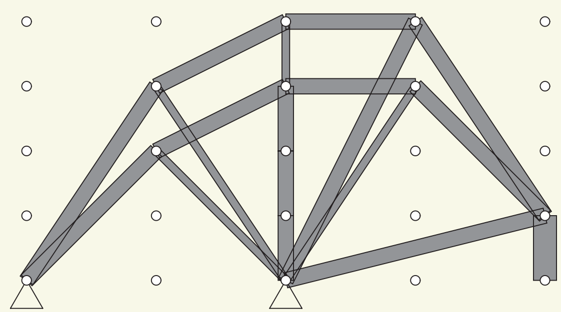


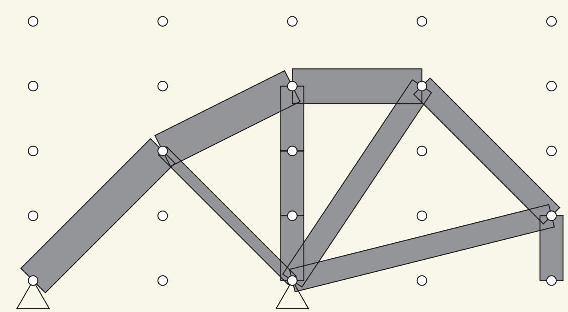
$x_e \in \{0, 1100\}$



$x_e \in \{0, 700, 1400\}$



$x_e \in \{0, 600, 1200, 1800\}$

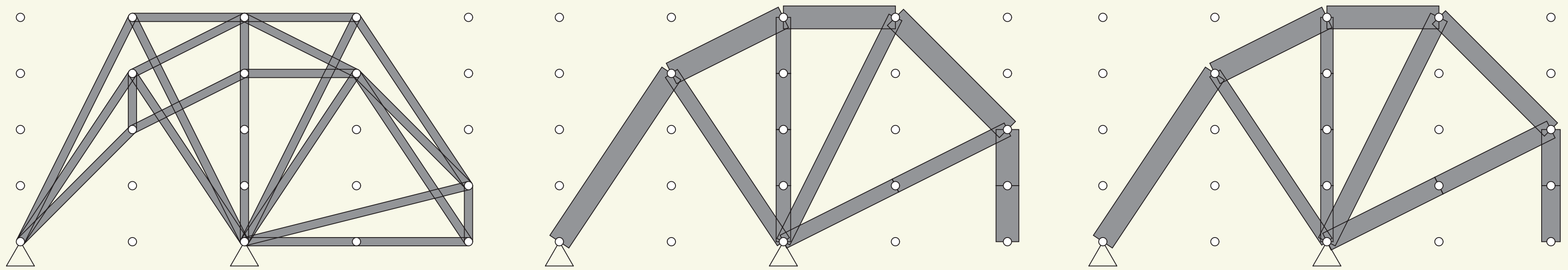


$x_e \in \{0, 400, 800, 1200\}$

- Choose c-s areas from a set of predetermined values.
- The # of different c-s areas is (certainly) small, but...
  - ...the opt. sol. highly depends on the predetermined values.

ex.) 200-bar truss: discrete optimization

- $x_e \in \{0, 120, 240, \dots, 1875\}$: 16 candidate values
  - w/ cstr. on # of different c-s areas

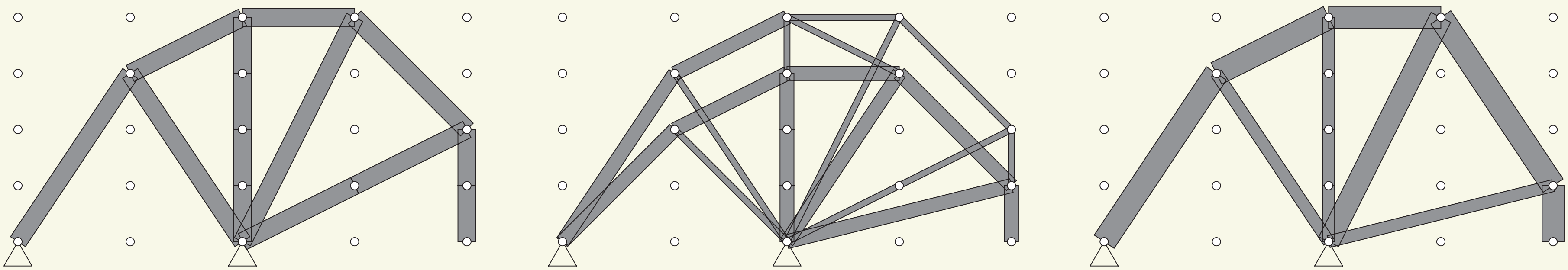


$n = 1$ $n = 2$ $n = 3$

n	Time (s)	
1	34.7	
2	68.3	
3	502.3	(< 9 min)

ex.) 200-bar truss: discrete optimization

- $x_e \in \{0, 125, 250, \ldots, 1800\}$: 16 candidate values
  - w/ cstr. on # of different c-s areas



$n = 1$ $n = 2$ $n = 3$

n	Time (s)	
1	22.8	
2	139.3	
3	1113.2	(< 20 min)

conclusions

- constraint on # of different design variables
  - Determine optimal values used for c-s areas, $y_1, \dots, y_n$.
    [continuous]
  - Determine which to choose among $0, y_1, \dots, y_n$ for member e.
    [combinatorial]
  - $\rightarrow$ linear ineq. with 0-1 variables
- compliance optimization of a truss
  - MISOCP (mixed-integer second-order cone programming)
  - integer variables + nonlinear convex constraints
  - global optimization
    - a branch-and-cut method (existing solvers)
  - For discrete optimization,...
    - MISOCP is more efficient than MILP/MIQP.