Symmetry of the Solution of Semidefinite Program by Using Primal-Dual Interior-Point Method

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Semi-Definite Programming (SDP)

1. convex programming
2. including LP, QP, etc.
3. primal-dual interior-point method (IPM)
   - polynomial time convergence
   - practical and fast softwares
4. application
   - system and control
   - combinatorial optimization
   - truss optimization
     - fundamental frequency (Ohsaki *et al.*, 1999)
     - linear buckling loads (Kanno *et al.*, 2000)
     - compliance (Ben-Tal and Nemirovski, 1997)

The standard form of SDP:

$$\begin{aligned}
\mathcal{P}: \ & \text{min} \ \ \boldsymbol{C} \bullet \boldsymbol{X} \\
& \text{s.t.} \ \ \boldsymbol{F}_i \bullet \boldsymbol{X} = b_i \quad (i = 1, \cdots, m), \\
& \qquad \boldsymbol{X} \in \mathcal{S}^n_+; \\
\mathcal{D}: \ & \text{max} \ \sum_{i=1}^{m} b_i y_i \\
& \text{s.t.} \ \ \sum_{i=1}^{m} \boldsymbol{F}_i y_i + \boldsymbol{Z} = \boldsymbol{C}, \\
& \qquad \boldsymbol{Z} \in \mathcal{S}^n_+.
\end{aligned}$$

Definitions

$\boldsymbol{U} \in \mathcal{S}^n \iff \boldsymbol{U}$ is a real $n \times n$ symmetric matrix

$\boldsymbol{U} \in \mathcal{S}^n_+ \iff \boldsymbol{U} \in \mathcal{S}^n$ is positive semidefinite

$\boldsymbol{U} \in \mathcal{S}^n_{++} \iff \boldsymbol{U} \in \mathcal{S}^n$ is positive definite

Variable matrices and vector

$$\boldsymbol{X} \in \mathcal{S}^n, \quad \boldsymbol{y} \in \Re^m, \quad \boldsymbol{Z} \in \mathcal{S}^n$$

Constant matrices and vector

$$\boldsymbol{b} \in \Re^m, \quad \boldsymbol{C} \in \mathcal{S}^n, \quad \boldsymbol{F}_i \in \mathcal{S}^n$$

Inner product

$$\boldsymbol{U} \bullet \boldsymbol{V} = \text{Tr}(\boldsymbol{U}^\top \boldsymbol{V}) = \sum_{i=1}^{n} \sum_{j=1}^{n} U_{i,j} V_{i,j}$$

Optimization for Specified Fundamental Frequency

Ground structure method:

- A truss with fixed location of nodes and members.

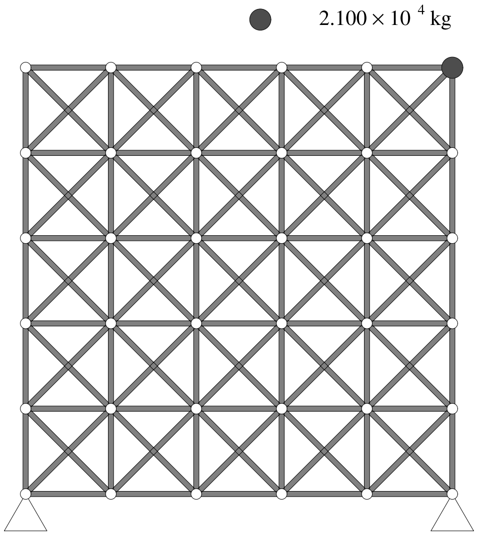


Fig. 1: Initial truss.

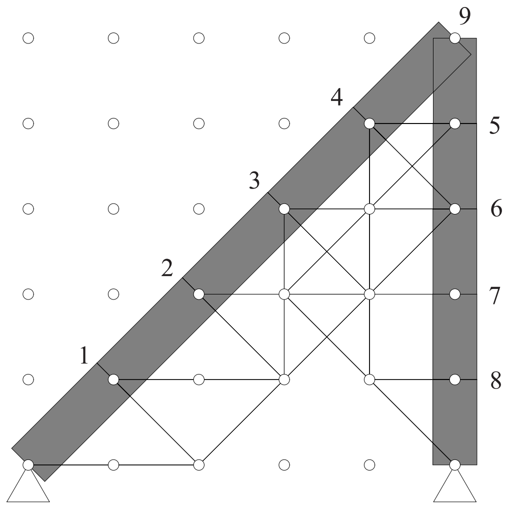


Fig. 2: Optimal solution.

Eigenvalue of free vibration Ω_r:

$$\boldsymbol{K}\boldsymbol{\Phi}_r = \Omega_r(\boldsymbol{M}_s + \boldsymbol{M}_0)\boldsymbol{\Phi}_r \quad (r = 1, 2, \cdots, n).$$

$\boldsymbol{K}$: linear stiffness matrix

$\boldsymbol{M}_s$: mass matrix (structural mass)

$\boldsymbol{M}_0$: mass matrix (nonstructural mass)

Optimization Problem:

$$
\begin{aligned}
\text{min} \quad & \sum_{i=1}^{N^m} b_i y_i \\
\text{s.t.} \quad & \Omega_r \geq \bar{\Omega}, \quad (r = 1, 2, \cdots, n), \\
& y_i \geq \underline{y}_i, \quad (i = 1, 2, \cdots, m).
\end{aligned}
$$

$\boldsymbol{y} = (y_i)$: member cross-sectional areas

$\boldsymbol{b} = (b_i)$: member lengths

$\bar{\Omega}$: specified fundamental eigenvalue

SDP formulation

$$
\begin{aligned}
\mathcal{D}' : \ \text{max} \quad & -\sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \quad & \sum_{i=1}^{m} (\boldsymbol{K}_i - \bar{\Omega}\boldsymbol{M}_i) y_i + \boldsymbol{Z} = -\bar{\Omega}\boldsymbol{M}_0, \\
& \boldsymbol{Z} \in \mathcal{S}_+^n, \ y_i \geq \underline{y}_i \quad (i = 1, 2, \cdots, m).
\end{aligned}
$$

$\boldsymbol{K}_i, \ \boldsymbol{M}_i$: constant matrices

Backgrounds:

1. Truss optimization
   (a) symmetric configuration
   (b) optimize cross-sectional areas $\boldsymbol{y}$
2. Question
   (a) Is the symmetric optimal $\bar{\boldsymbol{y}}$ always obtained?
   (b) symmetry: $\bar{y}_1 = \bar{y}_7$, $\bar{y}_2 = \bar{y}_6$ and $\bar{y}_3 = \bar{y}_5$?
3. Experimental results (Ohsaki *et al.*, 1999):
   (a) solution by IPM is symmetric.
   (b) solution by SQP is not symmetric.

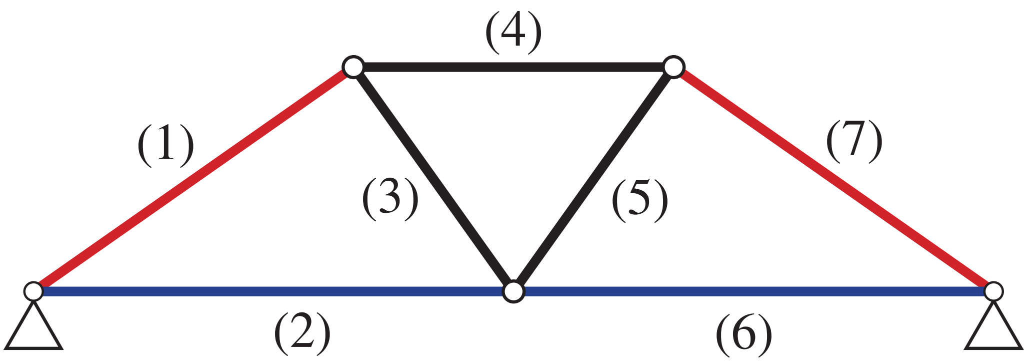


Fig. 3: A symmetric plane truss.

Our aim:

1. Theoretical proof
   (a) Definition of symmetric SDP
   (b) Symmetry of the central path (CP_μ)
   (c) Symmetry of the solution by IPM
2. Application—truss optimization

Generalized concept of symmetry

$x^P = x \Longleftrightarrow x$ is symmetry w.r.t. P

Definitions

1. $S(\Pi_n)$ for a vector $\boldsymbol{p} = \{p_i\} \in \Re^n$ as

$$p_i^{S(\Pi_n)} = p_{\Pi_n(i)}.$$

2. $Q(\Pi_n, \boldsymbol{e})$ for a matrix $\boldsymbol{A} = [A_{i,j}] \in \Re^{n \times n}$ as

$$A_{i,j}^{Q(\Pi_n,\boldsymbol{e})} = A_{\Pi_n(i),\Pi_n(j)} e_i e_j.$$

$$\Pi_n = \{\Pi_n(i) | i = 1, 2, \cdots, n\}$$

: a permutation of n indices $1, 2, \cdots, n$

$\boldsymbol{e} = (e_i) \in \Re^n$: $e_i = 1$ or -1.

Symmetry of solution

- For $\Pi_m = 7\ 6\ \cdots\ 2\ 1$,

$$\boldsymbol{y} = (y_1, y_2, \cdots, y_6, y_7),$$
$$\boldsymbol{y}^S(\Pi_m) = (y_7, y_6, \cdots, y_2, y_1).$$

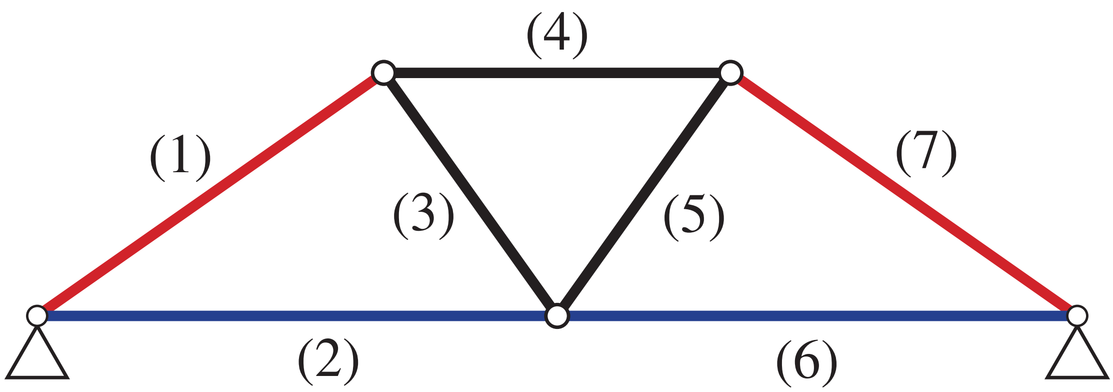


Fig. 4: A symmetric plane truss.

Symmetric solution:

- optimal solution $\bar{\boldsymbol{y}}$ satisfying $\bar{\boldsymbol{y}}^S = \bar{\boldsymbol{y}}$.

Standard form of SDP:

$$\mathcal{P}: \ \min \ \boldsymbol{C} \bullet \boldsymbol{X}$$
$$\text{s.t.} \ \ \boldsymbol{F}_i \bullet \boldsymbol{X} = b_i \ \forall i, \ \ \boldsymbol{X} \in \mathcal{S}^n_+;$$
$$\mathcal{D}: \ \max \ \sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \ \ \sum_{i=1}^{m} \boldsymbol{F}_i y_i + \boldsymbol{Z} = \boldsymbol{C}, \ \ \boldsymbol{Z} \in \mathcal{S}^n_+.$$

Symmetric SDP:

There exist Π_m, Π_n and $\boldsymbol{e} \in \Re^n$ such that

$$\boldsymbol{b}^{S(\Pi_m)} = \boldsymbol{b},$$
$$\boldsymbol{C}^{Q(\Pi_n, \boldsymbol{e})} = \boldsymbol{C},$$
$$\boldsymbol{F}_i^{Q(\Pi_n, \boldsymbol{e})} = \boldsymbol{F}_{\Pi_m(i)}.$$

$$\mathcal{D}': \ \max \quad -\sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \quad \sum_{i=1}^{m} (\boldsymbol{K}_i - \bar{\Omega}\boldsymbol{M}_i) y_i + \boldsymbol{Z} = -\bar{\Omega}\boldsymbol{M}_0,$$
$$\boldsymbol{Z} \in \mathcal{S}_+^n, \ y_i \geq \underline{y}_i \quad (i = 1, 2, \cdots, m).$$

Symmetry of $\mathcal{D}'$

— show $\boldsymbol{b}^S = \boldsymbol{b}$ and $\boldsymbol{K}_i^Q = \boldsymbol{K}_{\Pi_m(i)}$

- $\boldsymbol{b}$: member lengths

$$\boldsymbol{b} = (b_1, b_2, \cdots, b_6, b_7),$$
$$\boldsymbol{b}^S = (b_7, b_6, \cdots, b_2, b_1), \quad \text{for } \Pi_m = 8\ 7\ \cdots\ 2\ 1.$$

- configuration is symmetric

$\Longrightarrow \boldsymbol{b}^S = \boldsymbol{b}$

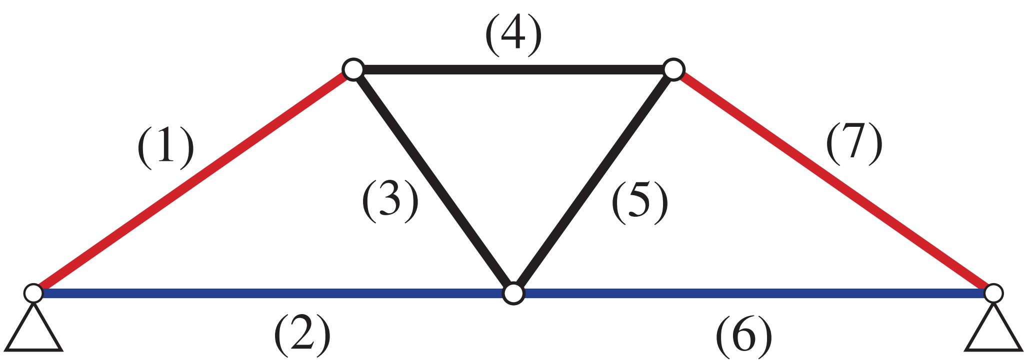


Fig. 5: A symmetric plane truss.

Symmetry of member stiffness matrices $\boldsymbol{K}_i$:

$$\boldsymbol{K}_1 = \frac{E}{b_1}\begin{bmatrix} 3/4 & \sqrt{3}/4 & 0 & 0 & 0 & 0 \\ \sqrt{3}/4 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Pi_n = 6\ 5\ 4\ 3\ 2\ 1$$

$$\boldsymbol{K}_7 = \frac{E}{b_7}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & \sqrt{3}/4 \\ 0 & 0 & 0 & 0 & \sqrt{3}/4 & 3/4 \end{bmatrix} \qquad \boldsymbol{K}_7^Q = \boldsymbol{K}_1$$

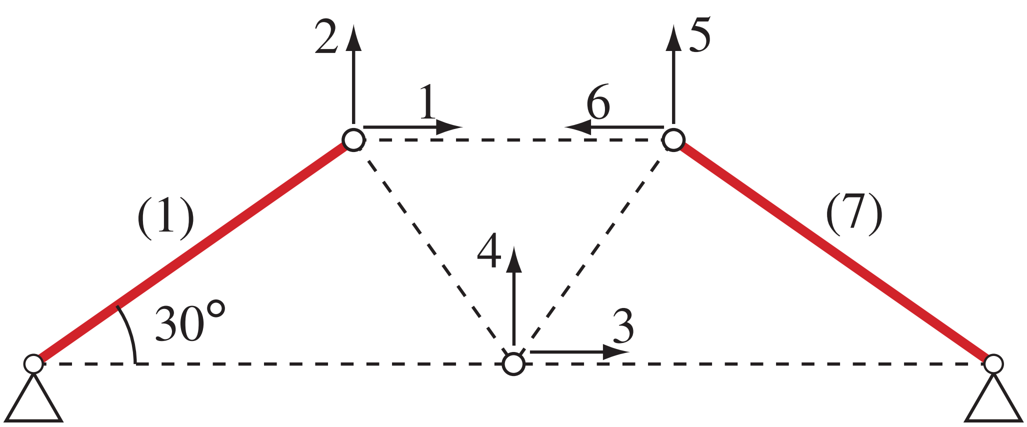


Fig. 6: A symmetric truss.

Central path of SDP

$$
\begin{array}{ll}
(\mathrm{CP}_\mu) & \boldsymbol{X}\boldsymbol{Z} = \mu \boldsymbol{I}, \qquad (\mu > 0) \\
& \boldsymbol{F}_i \bullet \boldsymbol{X} = b_i \ (i = 1, 2, \cdots, m), \ \boldsymbol{X} \in \mathcal{S}^n_{++}, \\
& \sum_{i=1}^{m} \boldsymbol{F}_i y_i + \boldsymbol{Z} = \boldsymbol{C}, \ \boldsymbol{Z} \in \mathcal{S}^n_{++}.
\end{array}
$$

$$
\Gamma = \{(\boldsymbol{X}(\mu), \boldsymbol{y}(\mu), \boldsymbol{Z}(\mu)) : \mu > 0\}
$$

1. continuous and smooth path in the feasible region.
2. converge to the optimal solution as $\mu \to 0$.
3. IPM computes $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{Z}})$ by tracing (CP_μ) as $\mu \to 0$.

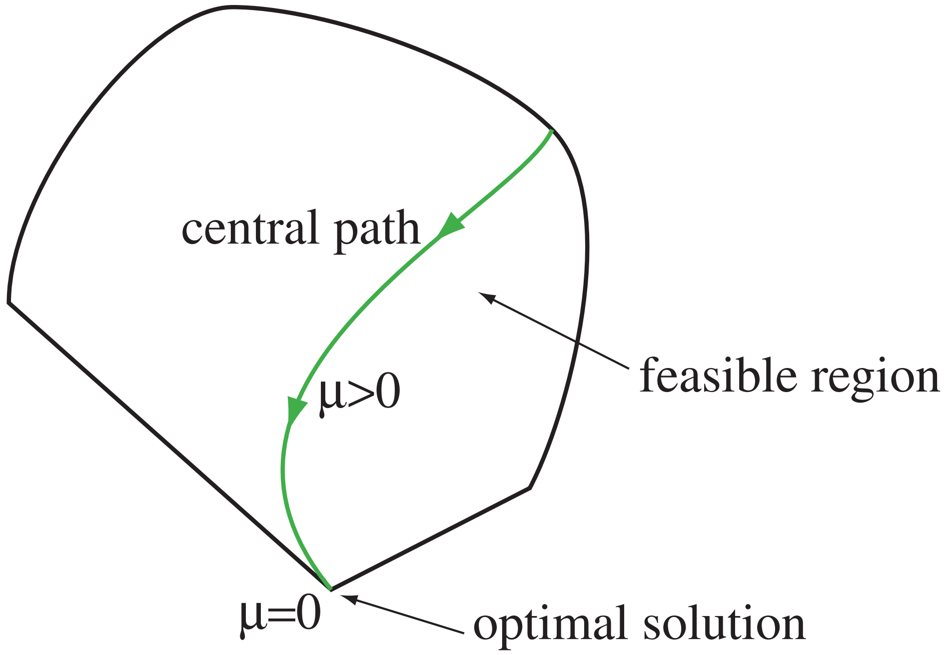


Fig. 7: Central path.

Theorem—symmetry of the central path:

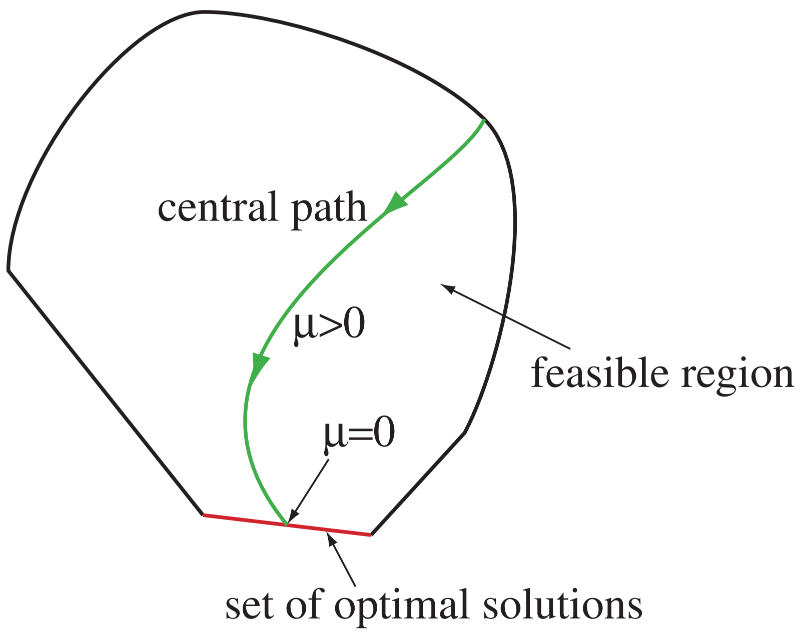
If the SDP problem is symmetric,
then $(\boldsymbol{X}(\mu), \boldsymbol{y}(\mu), \boldsymbol{Z}(\mu)) \in (\mathrm{CP}_\mu)$ is symmetric.

$$(\bar{\boldsymbol{X}}^Q, \bar{\boldsymbol{y}}^S, \bar{\boldsymbol{Z}}^Q) = (\bar{\boldsymbol{X}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{Z}}).$$

Outline of proof.

1. fix $\mu = \mu^*$.
2. suppose $(\boldsymbol{X}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*) \in (\mathrm{CP}_{\mu^*})$.
3. then, $(\boldsymbol{X}^{*Q}, \boldsymbol{y}^{*S}, \boldsymbol{Z}^{*Q}) \in (\mathrm{CP}_{\mu^*})$ is obtained.
4. from the uniqueness of the solution to (CP_{μ^*}), $(\boldsymbol{X}^{*Q}, \boldsymbol{y}^{*S}, \boldsymbol{Z}^{*Q}) = (\boldsymbol{X}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*)$ is obtained.

- $\Longrightarrow$ optimal solution $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{Z}}) = (\boldsymbol{X}(0), \boldsymbol{y}(0), \boldsymbol{Z}(0))$ by IPM is symmetric.



Examples: a 5-DOF truss

- Compare
  - IPM (primal-dual Interior-Point Method)
  - SQP (Sequential Quadratic Programming)
- Elastic modulus: 205.8 GPa, density: 7.86×10^{-3} kg
- $\bar{\Omega} = 1000$ rad^2/s^2, $\underline{y}_i = 10.0\ \mathrm{cm}^2$.
- An initial solution $\boldsymbol{y}^0$ is not symmetric.

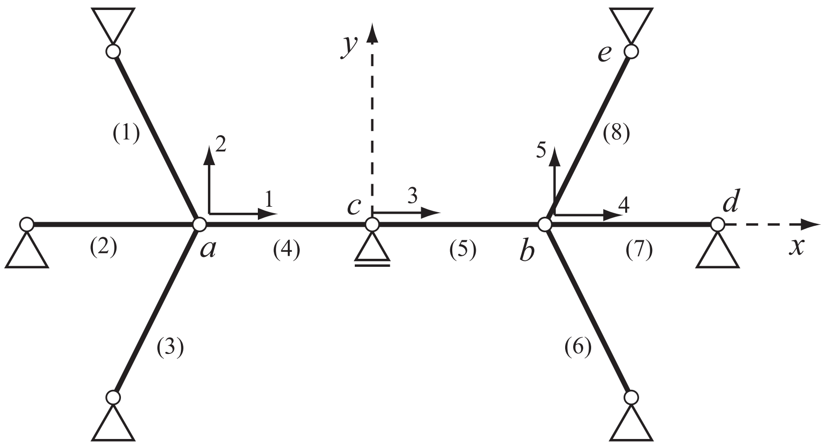


Fig. 8: Symmetric truss.

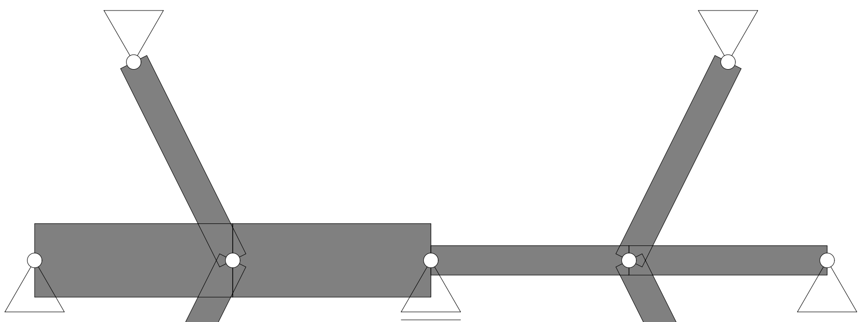


Fig. 9: Initial solution.

Examples: a 5-DOF truss

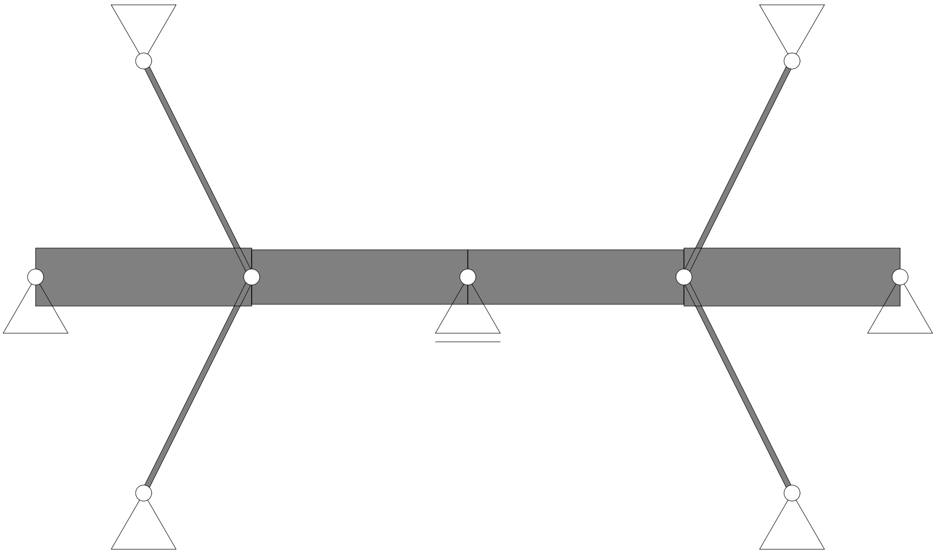


Fig. 10: Symmetric solution by IPM.

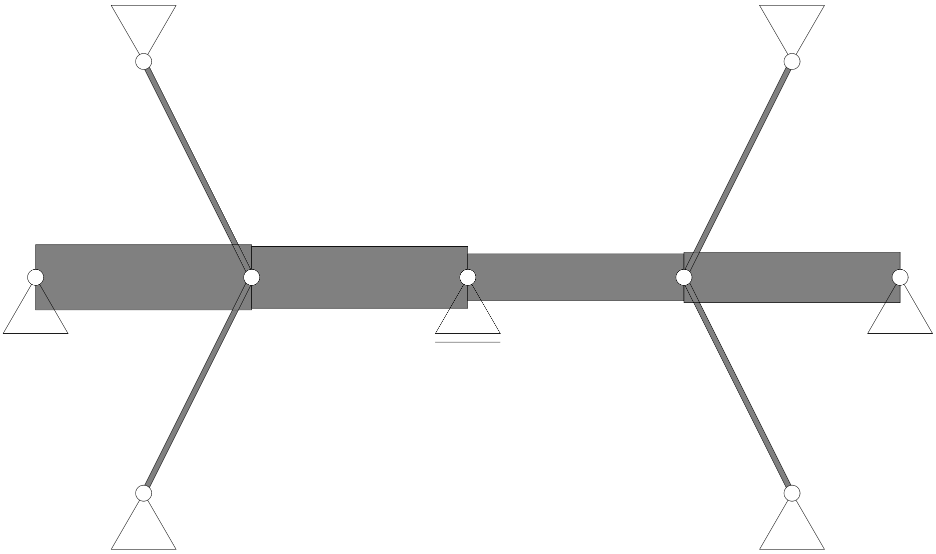


Fig. 11: Asymmetric solution by SQP.

	IPM	SQP
Vol. (cm^3)	46615.9	46615.9

Examples: a plane grid arch

algorithm	solution	accuracy	Vol. (cm^3)
IPM	symmetry	(6 digits)	774493.1
SQP	not symmetry	(2 digits)	774592.9

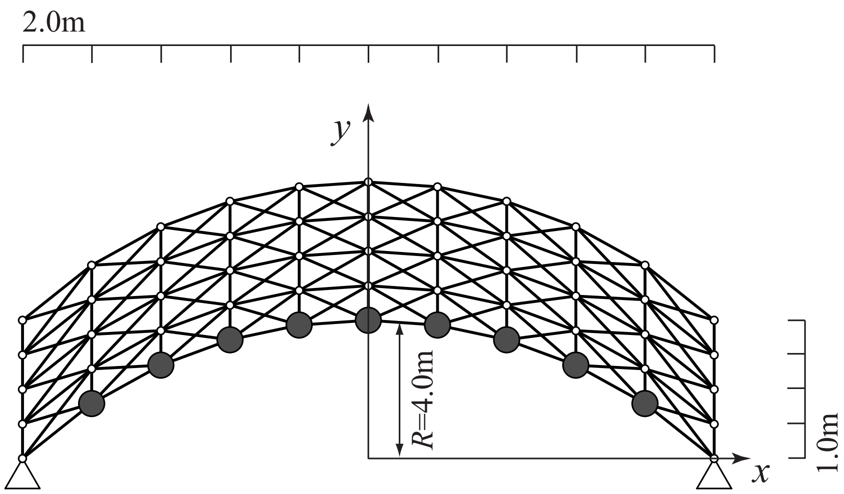


Fig. 12: A plane circular arch grid.

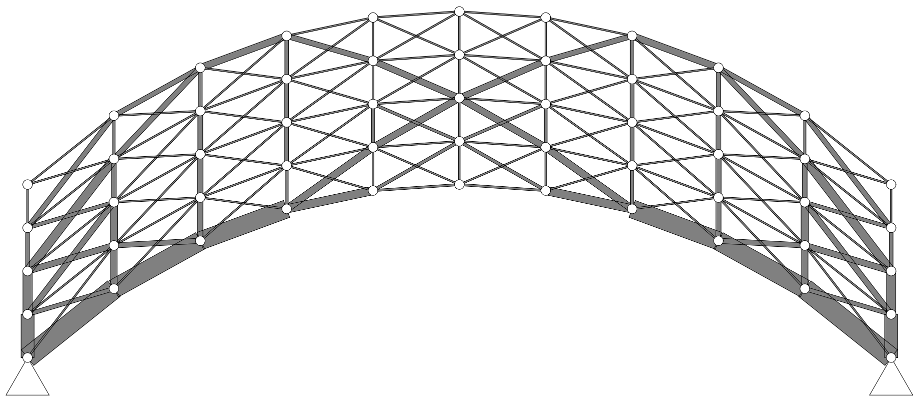


Fig. 13: Symmetric solution by IPM.

Conclusions

1. Symmetric SDP has been defined.
2. Symmetry of the central path has been proved.
3. The optimal solution obtained by a primal-dual interior-point method is always symmetric.
4. Eigenvalue optimization problem of a symmetric truss configuration has been formulated as symmetric SDP.
5. The symmetric solution can be obtained by IPM, where
   - there exists the other optimal solution that is not symmetric.
   - the conventional nonlinear programming approach converges to a solution that is not symmetric.