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Direct Evaluation of Robustness Functions of Trusses associated with Stress Constraints

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Abstract A direct computational method is presented for the evaluation of the robustness function of linear elastic trusses associated with stress constraints. Under the uncertainties of external forces based on the info-gap model, the robustness function is formulated as the optimal objective value of an optimization problem with infinitely many constraint conditions. By using the strong duality theory of the second-order cone programming problem, we reformulate the present problem to a numerically tractable form without any variable. The robustness functions are computed for various trusses under several uncertain loading circumstances.

Keywords: robustness; info-gap model; second-order cone program; duality; data uncertainty; reliability

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1. Introduction.

In structural and mechanical design, many conventional deterministic design optimization models have been successfully developed. Recently, the robust optimal design has received increasing attention, which may yield a system response that is less sensitive to uncertainties of various kinds of parameters. A number of reliability-based optimization methods as well as the robust optimal design methods have been developed for the structural and mechanical designs [5, 11–14, 18, 22–24].

Based on the stochastic uncertainty model of mechanical parameters, various methods were proposed for reliability-based optimization. The structural optimization by minimizing the failure probability was studied, where the failure probability was estimated by using the Monte-Carlo simulation [22, 26] with the response-surface approximation [24]. In order to reduce the computational cost in the evaluation of the failure probability, the reliability index approach was utilized [12, 18]. Various formulations for sensitivity analysis of probabilistic constraints were also proposed [9, 14]. Doltsinis and Kang [11] performed the multi-objective optimization so as to minimize both the expected value and the standard deviation of the goal performance.

On the other hand, as a non-probabilistic but bounded uncertainty model, the so-called convex model has been developed [3]. A unified methodology of *robust counterpart* of the various convex optimization problems was reviewed by Ben-Tal and Nemirovski [7]. Calafiore and El Ghaoui [8] proposed a method for finding the ellipsoidal bound of the solution set of uncertain linear equations based on the semidefinite programming relaxation. For further application of the robust optimization, see [4, 5]. In the field of structural optimization, Pantelides and Ganzerli [23] proposed a truss optimization method by using the convex model for uncertainties. Han and Kwak [13] attempted to find the design which minimizes the infinity norm of a vector of sensitivity coefficients of the performance functions.

Recently, based on the *info-gap decision theory*, the concept of *robustness function* was proposed by Ben-Haim [2]. The robustness function expresses the degree of the immunity to failure, or, the greatest level of non-probabilistic uncertainty at which any constraint in a mechanical system cannot be violated. The robustness function has the advantage, compared with the reliability based on a stochastic uncertainty model, such that engineers do not have to estimate neither the level of uncertainty nor the probabilistic distribution of uncertain parameters in order to evaluate the robustness of a structure, i.e., the robustness function does not require any information on statistical variation of the design parameters, which is often difficult to obtain practically. However, it is difficult to compute the robustness functions in the sense that we have to solve an optimization problem with infinitely many constraint conditions. This prevents us from applying the info-gap decision theory [2] to the reliability design of structural and mechanical systems. Hence, it is strongly desired to develop efficient methods for computing the robustness functions.

In this paper, we propose a computable reformulation of robustness functions of linear elastic trusses associated with stress constraints. Under the uncertainties of external forces, it is shown that the robustness functions can be exactly found, where the computational costs are bounded by the polynomial of the dimensions of system and uncertainty parameters.

Our approach in this paper is summarized as follows. We first show that the robustness function can be obtained as the optimal objective value of an optimization problem with finite number of variables and infinitely many constraint conditions. Secondly, by using the strong duality theory of the *second-order cone programming* (*SOCP*) problem [1], we reformulate the present problem to a numerically tractable form without any variable.

This paper is organized as follows. In Section 2, in order to make this paper self-contained, we introduce the concept of robustness functions proposed in [2], and the strong duality theorem of SOCP [1]. Section 3 investigates a simple truss example analytically in order to explain the engineering meaning of robustness functions more clearly. In Section 4, we formulate a mathematical programming problem with some variables and infinitely many constraint conditions such that the optimal objective value coincides with the robustness function of trusses associated with stress constraints. A tractable reformulation of the present problem is proposed in Section 5 by using the strong duality of SOCP. In Section 6, the robustness functions of trusses with various uncertainty sets are obtained by using the proposed formulations.

2. Preliminaries.

Throughout the paper, all vectors are assumed to be column vectors. However, for vectors $\boldsymbol{p} \in \mathbf{R}^n$ and $\boldsymbol{q} \in \mathbf{R}^m$, we often write

$$(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}^\top, \boldsymbol{q}^\top)^\top \in \mathbf{R}^{n+m}$$

in order to simplify the notation. The standard Euclidean norm $\|\boldsymbol{p}\|_2 = (\boldsymbol{p}^\top \boldsymbol{p})^{1/2}$ of a vector $\boldsymbol{p} \in \mathbf{R}^n$ is often abbreviated by $\|\boldsymbol{p}\|$. $\|\boldsymbol{p}\|_\infty$ and $\|\boldsymbol{p}\|_1$, respectively, denote the l_∞-norm and l_1-norm of $\boldsymbol{p} = (p_i) \in \mathbf{R}^n$ defined as

$$\|\boldsymbol{p}\|_\infty = \max_{i \in \{1,\dots,n\}} |p_i|, \tag{1}$$

$$\|\boldsymbol{p}\|_1 = \sum_{i=1}^{n} |p_i|. \tag{2}$$

$\mathcal{S}^n \subset \mathbf{R}^{n \times n}$ denotes the set of all $n \times n$ real symmetric matrices.

2.1. Robustness function.

Let $\boldsymbol{x} \in \mathbf{R}^m$ denote the vector of design variables of a structure. $\boldsymbol{\zeta} \in \mathbf{R}^n$ denotes the parameter vector that is considered to be uncertain, or, inexact. For a given $\alpha \geq 0$, $\boldsymbol{\zeta}$ is assumed to be running through a given *uncertain set* $\mathcal{Z}(\alpha) = \{\boldsymbol{\zeta}\} \subset \mathbf{R}^n$. Throughout this paper, we make the following assumption:

Assumption 2.1.

(i) $\mathcal{Z}(0) = \{\mathbf{0}\}$;

(ii) *If* $0 \leq \alpha_1 < \alpha_2$, *then* $\mathcal{Z}(\alpha_1) \subset \mathcal{Z}(\alpha_2)$.

Roughly speaking, Assumption 2.1 implies that $\boldsymbol{\zeta} \in \mathcal{Z}(\alpha)$ perturbs around the origin with the 'width' of α. For a given $\boldsymbol{\zeta}^0 \in \mathcal{Z}(\alpha)$, suppose that the constraints on the structure are written as

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\zeta}^0) - r^{\mathrm{c}} \boldsymbol{q} \geq \mathbf{0}, \tag{3}$$

where $\boldsymbol{q} \in \mathbf{R}^k$ and $r^{\mathrm{c}} \in \mathbf{R}$ are constant, $g_i : \mathbf{R}^m \times \mathbf{R}^n \mapsto \mathbf{R}$, and $\boldsymbol{g} = (g_i)_{i=1}^k$. Then the robustness function $\widehat{\alpha} : \mathbf{R}^{m+1} \mapsto (-\infty, +\infty]$ associated with the constraints (3) is defined by (see, e.g., [2, Ch. 3])

$$\widehat{\alpha}(\boldsymbol{x}, r^{\mathrm{c}}) = \max\{\alpha : \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\zeta}) - r^{\mathrm{c}}\boldsymbol{q} \geq \mathbf{0} \ (\forall \boldsymbol{\zeta} \in \mathcal{Z}(\alpha))\}. \tag{4}$$

We write $\widehat{\alpha}(\boldsymbol{x}, r^{\mathrm{c}}) = 0$ if the feasible set of Problem (4) is empty. In what follows, $\widehat{\alpha}(\boldsymbol{x}, r^{\mathrm{c}})$ is often abbreviated by $\widehat{\alpha}$. For the two different sets of design variables $\boldsymbol{x}^1 \in \mathbf{R}^m$ and $\boldsymbol{x}^2 \in \mathbf{R}^m$, we say that $\boldsymbol{x}^1$ is *more robust* than $\boldsymbol{x}^2$ if $\widehat{\alpha}(\boldsymbol{x}^1, r^{\mathrm{c}}) > \widehat{\alpha}(\boldsymbol{x}^2, r^{\mathrm{c}})$. If $\boldsymbol{\zeta}^1 \in \mathcal{Z}(\widehat{\alpha}(\boldsymbol{x}^1, r^{\mathrm{c}}))$ satisfies

$$\exists i \in \{1, \dots, k\} \quad \text{s.t.} \quad g_i(\boldsymbol{x}^1, \boldsymbol{\zeta}^1) - r^{\mathrm{c}} q_i = 0,$$

then we say that $\boldsymbol{\zeta}^1$ is the *worst case.*

2.2. Strong duality theorem of second-order cone program.

Let $\mathbf{R}_+^n \subset \mathbf{R}^n$ and $\mathbf{L}_+^n \subset \mathbf{R}^n$ denote the non-negative orthant and the *second-order cone* (or *Lorentz cone*) [1], respectively, which are defined as

$$\begin{aligned}
\mathbf{R}_+^n &= \{\boldsymbol{p} = (p_1, \dots, p_n) \in \mathbf{R}^n | p_i \geq 0 \ (i = 1, \dots, n)\}, \\
\mathbf{L}_+^n &= \{\boldsymbol{p} = (p_0, \boldsymbol{p}_1) \in \mathbf{R}^1 \times \mathbf{R}^{n-1} | p_0 \geq \|\boldsymbol{p}_1\|_2\}.
\end{aligned}$$

Let

$$\mathcal{K}_i = \mathbf{R}_+^{k_i} \text{ or } \mathbf{L}_+^{k_i} \quad (i = 1, \dots, n),$$

and $k := \sum_{i=1}^n k_i$. The *second-order cone programming* (*SOCP*) problem [1] refers to the optimization problem having the form of

$$\left.\begin{aligned}
(\mathrm{P_{SOCP}}): \quad & \min \quad \sum_{i=1}^n \boldsymbol{c}_i^\top \boldsymbol{x}_i \\
& \text{s.t.} \quad \sum_{i=1}^n \boldsymbol{A}_i \boldsymbol{x}_i = \boldsymbol{b}, \quad \boldsymbol{x}_i \in \mathcal{K}_i \quad (i = 1, \dots, n),
\end{aligned}\right\} \tag{5}$$

where $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \mathbf{R}^{k_1} \times \dots \times \mathbf{R}^{k_n}$ is the variable, and $\boldsymbol{A}_i \in \mathbf{R}^{m \times k_i}$, $\boldsymbol{b} \in \mathbf{R}^m$, and $\boldsymbol{c}_i \in \mathbf{R}^{k_i}$ $(i = 1, \dots, n)$ are constant. The dual of Problem (5) is formulated in variables $\boldsymbol{y} \in \mathbf{R}^m$ as

$$\left.\begin{aligned}
(\mathrm{D_{SOCP}}): \quad & \max \quad \boldsymbol{b}^\top \boldsymbol{y} \\
& \text{s.t.} \quad \boldsymbol{c}_i - \boldsymbol{A}_i^\top \boldsymbol{y} \in \mathcal{K}_i \quad (i = 1, \dots, n).
\end{aligned}\right\} \tag{6}$$

SOCP has received increasing attention for its broad application [1, 6, 17]. The primal-dual interior-point methods, which were developed for LP [19] at first, have been naturally extended to SOCP [20].

Let $\mathrm{int}\,\mathcal{K}_i$ denote the interior of $\mathcal{K}_i$ (see, e.g., [25]) defined as

$$\begin{aligned}
\mathrm{int}\,\mathbf{R}_+^{n_i} &= \{\boldsymbol{p} = (p_1, \dots, p_{n_i}) \in \mathbf{R}^{n_i} | p_i > 0 \ (i = 1, \dots, n_i)\}, \\
\mathrm{int}\,\mathbf{L}_+^{n_i} &= \{\boldsymbol{p} = (p_0, \boldsymbol{p}_1) \in \mathbf{R}^1 \times \mathbf{R}^{n_i-1} | p_0 > \|\boldsymbol{p}_1\|_2\}.
\end{aligned}$$

We introduce the following assumptions on the primal-dual pair of SOCP problems:

Assumption 2.2.

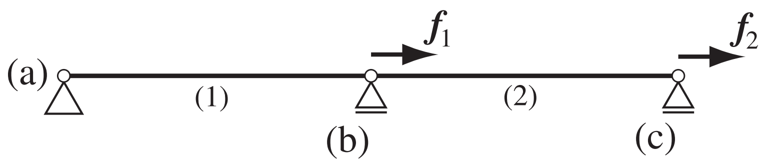

Figure 1: Two-degrees-of-freedom truss.

(i) *The rows of the matrix* $\widetilde{\boldsymbol{A}} := (\boldsymbol{A}_1, \dots, \boldsymbol{A}_n) \in \mathbf{R}^{m \times k}$ *are linearly independent;*

(ii) *Both* $(\mathrm{P_{SOCP}})$ *and* $(\mathrm{D_{SOCP}})$ *have strictly feasible solutions, i.e.,*

$$\left\{(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_n} \Big| \sum_{i=1}^{n} \boldsymbol{A}_i \boldsymbol{x}_i = \boldsymbol{b},\ \boldsymbol{x}_i \in \operatorname{int} \mathcal{K}_i\ (i = 1, \dots, n)\right\} \neq \emptyset,$$

$$\{\boldsymbol{y} \in \mathbf{R}^m | \boldsymbol{c}_i - \boldsymbol{A}_i^\top \boldsymbol{y} \in \operatorname{int} \mathcal{K}_i\ (i = 1, \dots, n)\} \neq \emptyset$$

are satisfied.

$(\mathrm{P_{SOCP}})$ and $(\mathrm{D_{SOCP}})$ are known to satisfy the following duality property:

Theorem 2.3 (Strong duality of SOCP). *Under Assumption 2.2,*

(i) $(\mathrm{P_{SOCP}})$ *and* $(\mathrm{D_{SOCP}})$ *have the optimal solutions* $(\overline{\boldsymbol{x}}_1, \dots, \overline{\boldsymbol{x}}_n)$ *and* $\overline{\boldsymbol{y}}$*, respectively, and*

$$\sum_{i=1}^{n} \boldsymbol{c}_i^\top \overline{\boldsymbol{x}}_i = \boldsymbol{b}^\top \overline{\boldsymbol{y}}. \tag{7}$$

(ii) *feasible solutions* $(\overline{\boldsymbol{x}}_1, \dots, \overline{\boldsymbol{x}}_n)$ *and* $\overline{\boldsymbol{y}}$ *of* $(\mathrm{P_{SOCP}})$ *and* $(\mathrm{D_{SOCP}})$*, respectively, are the optimal solutions if and only if* (7) *is satisfied.*

Proof. See Theorem 2.4.1 in [6]. □

Theorem 2.3 plays a fundamental role in a tractable formulation of robustness functions in Section 5. For engineering use of the strong duality of SOCP, see [15, 16].

3. Illustrative example.

Consider a linear elastic two-degrees-of-freedom truss as illustrated in Fig.1. In order to introduce the concept of robustness functions in more detail, this section illustrates how to find the robustness functions analytically for the simple truss example.

Node (a) in Fig.1 is pin-supported, whereas the external forces f_1 and f_2 are applied at nodes (b) and (c), respectively. u_1 and u_2 denote the displacements of nodes (b) and (c), respectively. The member stresses σ_i are obtained as

$$\sigma_1 = \frac{f_1 + f_2}{a_1}, \quad \sigma_2 = \frac{f_2}{a_2},$$

where a_i denotes the cross-sectional area of the ith member. Consider the stress constraints written as

$$|\sigma_i| \leq \sigma^{\mathrm{c}} \quad (i = 1, 2), \tag{8}$$

where $\sigma^{\rm c} > 0$ is a constant.

Suppose that (f_1, f_2) may deviate from the nominal value, where its uncertainty can be expressed by an *interval info-gap model* [2] defined as

$$f_1 = f_2 = \widetilde{f} + \zeta, \quad \alpha \geq |\zeta|, \quad \alpha \geq 0. \tag{9}$$

Here, $\widetilde{f} > 0$ denotes the nominal value of f_1 and f_2, and ζ the parameter representing the uncertainty. The robustness function $\widehat{\alpha}(\boldsymbol{a}, \sigma^{\rm c})$ is defined as the greatest value of α for which the constraints (8) are not violated with any (f_1, f_2) satisfying (9). It follows from $\widetilde{f} > 0$ that the maximum absolute value of stresses are obtained as

$$\max_{\zeta \in \mathbf{R}} \{|\sigma_1| : \alpha \geq |\zeta|\} = \frac{2(\widetilde{f} + \alpha)}{a_1}, \quad \max_{\zeta \in \mathbf{R}} \{|\sigma_2| : \alpha \geq |\zeta|\} = \frac{\widetilde{f} + \alpha}{a_2}. \tag{10}$$

Let $\widetilde{f} = 1.0$ and $\sigma^{\rm c} = 0.2$. Suppose that members (1) and (2) have the same unstressed member lengths. Consider the following two cases:

$$\boldsymbol{a}^1 = (15, 15)^\top, \quad \boldsymbol{a}^2 = (20, 10)^\top,$$

which have the same structural volume. First, let $\boldsymbol{a} = \boldsymbol{a}^1$. It follows from (10) that $\max\{|\sigma_1| : \alpha \geq |\zeta|\}$ attains $\sigma^{\rm c}$ at $\alpha = 0.5$. On the other hand, $\max\{|\sigma_2| : \alpha \geq |\zeta|\}$ attains $\sigma^{\rm c}$ at $\alpha = 2.0$. Consequently, we obtain

$$\widehat{\alpha}(\boldsymbol{a}^1, \sigma^{\rm c}) = 0.5.$$

The worst case corresponds to $f_1 = f_2 = 1.5$, where the stress constraint of member (1) becomes active in tension state. $\widehat{\alpha} = 0.5$ implies that the member stresses are guaranteed to be acceptable in the sense of (8), if the uncertain external forces satisfy (9) with $\alpha = 0.5$. Next, letting $\boldsymbol{a} = \boldsymbol{a}^2$, we see for each $i = 1, 2$ that

$$\max_{\zeta \in \mathbf{R}} \{|\sigma_i| : \alpha \geq |\zeta|\} = \sigma^{\rm c} \quad \Longleftrightarrow \quad \alpha = 1.0,$$

from which we obtain

$$\widehat{\alpha}(\boldsymbol{a}^2, \sigma^{\rm c}) = 1.0.$$

The worst case corresponds to $f_1 = f_2 = 2.0$, where the stress constraints of members (1) and (2) become active in tension state. Hence, we conclude that the robustness function of the design $\boldsymbol{a}^2$ is larger than that of $\boldsymbol{a}^1$, i.e., $\boldsymbol{a}^2$ is more robust than $\boldsymbol{a}^1$.

Unfortunately, if the mechanical system has moderately many degrees of freedom and/or the uncertainty set has a complicated structure, it is difficult to find the worst case perturbation and the corresponding active constraint conditions. This is the crucial difficulty in evaluating the robustness function. In the following section, we propose a unified formulation for calculating the robustness function. A numerically tractable reformulation scheme is also proposed in Section 5.

4. Robustness functions associated with stress constraints.

In this section, we show that the robustness function of trusses associated with stress constraints can be obtained as the optimal objective value of a mathematical programming problem with infinitely many constraint conditions.

Consider a linear elastic truss in the three-dimensional space. Let n^{d} denote the number of degrees of freedom of displacements. $\boldsymbol{u} \in \mathbf{R}^{n^{\mathrm{d}}}$ and $\boldsymbol{f} \in \mathbf{R}^{n^{\mathrm{d}}}$ denote the vectors of nodal displacements and external forces, respectively. The system of equilibrium equations can be written as

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}, \tag{11}$$

where $\boldsymbol{K} \in \mathcal{S}^{n^{\mathrm{d}}}$ denotes the stiffness matrix of the truss. In what follows, we suppose that $\boldsymbol{f}$ in (11) has the bounded uncertainty (rigorously defined below).

Let $\boldsymbol{a} = (a_i) \in \mathbf{R}^{n^{\mathrm{m}}}$ denote the vector of cross-sectional areas, where n^{m} denotes the number of members. Observing that the stiffness matrix of the truss satisfies

$$\operatorname{rank}(\partial \boldsymbol{K}/\partial a_i) = 1 \quad (i = 1, \dots, n^{\mathrm{m}}),$$

we can write

$$\boldsymbol{K}(\boldsymbol{a}) = \sum_{i=1}^{n^{\mathrm{m}}} a_i \boldsymbol{b}_i \boldsymbol{b}_i^{\top}. \tag{12}$$

Here, for each $i = 1, \dots, n^{\mathrm{m}}$, $\boldsymbol{b}_i = (b_{ij}) \in \mathbf{R}^{n^{\mathrm{d}}}$ is a constant vector.

Consider the constraints on nodal displacements $\boldsymbol{u}$ formulated as

$$\|\boldsymbol{D}\boldsymbol{u}\|_{\infty} \leq u^{\mathrm{c}}, \tag{13}$$

or, equivalently,

$$|\boldsymbol{d}_l^{\top}\boldsymbol{u}| \leq u_l^{\mathrm{c}} \quad (l = 1, \dots, n^{\mathrm{c}}),$$

where $\mathbf{R} \ni u^{\mathrm{c}} > 0$, $\mathbf{R} \ni u_l^{\mathrm{c}} \geq 0$, $\boldsymbol{D} \in \mathbf{R}^{n^{\mathrm{c}} \times n^{\mathrm{d}}}$, and $\boldsymbol{d}_l \in \mathbf{R}^{n^{\mathrm{d}}}$ are constant. Here, from the definition (1) of l_{∞}-norm, we immediately have

$$\|\boldsymbol{D}\boldsymbol{u}\|_{\infty} = \max_{k \in \{1, \dots, n^{\mathrm{c}}\}} |(\boldsymbol{D}\boldsymbol{u})_k|,$$

where $(\boldsymbol{D}\boldsymbol{u})_k$ denots the kth component of the vector $\boldsymbol{D}\boldsymbol{u}$. Let E denote the elastic modulus. It follows from (12) that the stress constraints are written in a form of

$$\left|\sqrt{E}\boldsymbol{b}_i^{\top}\boldsymbol{u}\right| \leq \sigma_i^{\mathrm{c}} \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{14}$$

which can be easily embedded into (13).

Throughout the paper, we restrict ourselves to the cases where only the external forces $\boldsymbol{f}$ have uncertainty. Let $\tilde{\boldsymbol{f}} \in \mathbf{R}^{n^{\mathrm{d}}}$ denote the nominal value of $\boldsymbol{f}$. For the given $\tilde{\boldsymbol{f}}$ and $\alpha \geq 0$, we denote the uncertainty set of $\boldsymbol{f}$ by

$$\mathcal{T}(\alpha, \tilde{\boldsymbol{f}}) \subset \mathbf{R}^{n^{\mathrm{d}}},$$

which is given as the affine image of a set $\mathcal{Z}(\alpha)$ satisfying Assumption 2.1. In what follows, $\mathcal{T}(\alpha, \widetilde{\boldsymbol{f}})$ is often abbreviated by $\mathcal{T}(\alpha)$ or $\mathcal{T}$. Let $\mathcal{U}(\alpha) \subseteq \mathbf{R}^{n^{\rm d}}$ denote the set of all the possible solutions to (11), i.e.,

$$\mathcal{U}(\alpha) = \left\{ \boldsymbol{u} \in \mathbf{R}^{n^{\rm d}} \,|\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}, \text{ for some } \boldsymbol{f} \in \mathcal{T}(\alpha) \right\}. \tag{15}$$

In accordance with (4), the robustness function $\widehat{\alpha} : \mathbf{R}^{n^{\rm m}+1} \mapsto (-\infty, +\infty]$ associated with the stress constraints (13) is defined as

$$\widehat{\alpha}(\boldsymbol{a}, u^{\rm c}) = \max \left\{ \alpha : u^{\rm c} \geq \|\boldsymbol{D}\boldsymbol{u}\|_\infty \quad \forall \boldsymbol{u} \in \mathcal{U}(\alpha) \right\}. \tag{16}$$

Consequently, the robustness function $\widehat{\alpha}$ can be obtained by solving the optimization problem (16). However, it should be emphasized that Problem (16) is numerically intractable, because it has infinite number of constraints. This motivates us to investigate in Section 5 a reformulation scheme of Problem (16).

5. Direct evaluations of robustness functions.

We investigate in this section some particular cases of uncertainty sets $\mathcal{T}$, and propose a numerically tractable formulation to calculate the robustness function $\widehat{\alpha}$ concerning with the stress constraints.

Throughout this section, suppose

$$u^{\rm c} \geq \|\boldsymbol{D}\boldsymbol{u}\|_\infty \quad \forall \boldsymbol{u} \in \mathcal{U}(0), \tag{17}$$

otherwise we obtain $\widehat{\alpha}(\boldsymbol{a}, u^{\rm c}) = 0$ from the definition. We write $\boldsymbol{K} := \boldsymbol{K}(\boldsymbol{a})$ for simplicity.

We first investigate a relatively general setting of the uncertainty set. Let $\boldsymbol{\zeta}_p = (\zeta_{pq})_{q=1}^{n_p}$, $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_m) \in \mathbf{R}^{n^\zeta}$ and $n^\zeta = \sum_{p=1}^m n_p$. Assume that $\mathcal{T}$ is given as the affine image of a bounded set $\mathcal{Z}(\alpha) = \{\boldsymbol{\zeta}\} \subset \mathbf{R}^{n^\zeta}$; i.e.,

$$\mathcal{T}(\alpha, \widetilde{\boldsymbol{f}}) = \left\{ \widetilde{\boldsymbol{f}} + \sum_{p=1}^m \sum_{q=1}^{n_p} \zeta_{pq} \boldsymbol{f}^{pq} \middle| \boldsymbol{\zeta} \in \mathcal{Z}(\alpha) \right\}, \tag{18}$$

where $\boldsymbol{f}^{pq} \in \mathbf{R}^{n^{\rm d}}$ is constant for each $q = 1, \dots, n_p$ $(p = 1, \dots, m)$. Suppose

$$\mathcal{Z}(\alpha) = \{ \boldsymbol{\zeta} \in \mathbf{R}^{n^\zeta} | \alpha \geq \|\boldsymbol{\zeta}_p\|_2 \ (p = 1, \dots, m) \}. \tag{19}$$

Notice here that the definition (19) of $\mathcal{Z}(\alpha)$ is motivated by the assumption such that the uncertainty parameters $\zeta_{p1}, \dots, \zeta_{pn_p}$ perturb uncorrelatedly under the norm constraint, whereas $\zeta_{p_1 q_1}$ and $\zeta_{p_2 q_2}$ are independent if $p_1 \neq p_2$.

The following proposition states our main result:

Proposition 5.1. *Suppose that $\mathcal{T}$ and $\mathcal{Z}$ are given by* (18) *and* (19), *respectively. Then the robustness function $\widehat{\alpha}$ is explicitly obtained as*

$$\widehat{\alpha}(\boldsymbol{a}, u^{\rm c}) = \min_{l=1,\dots,n^{\rm c}} \left\{ \min \left\{ \frac{\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\rm c}}{\sum_{p=1}^m \left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^{pq} \right)_{q=1}^{n_p} \right\|_2}, \frac{-\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\rm c}}{\sum_{p=1}^m \left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^{pq} \right)_{q=1}^{n_p} \right\|_2} \right\} \right\}. \tag{20}$$

Proof. A point $(\alpha, \boldsymbol{u}) \in \mathbf{R} \times \mathbf{R}^{n^{\rm d}}$ is a feasible solution of Problem (16) if and only if $(\alpha, \boldsymbol{u})$ solves the following semi-infinite system:

$$
\left.
\begin{aligned}
&-\|\boldsymbol{D}\boldsymbol{u}\|_{\infty} + u^{\rm c} \geq 0 \\
&\forall \boldsymbol{u} \in \left\{ \boldsymbol{u} \in \mathbf{R}^{n^{\rm d}} \;\middle|\; \boldsymbol{K}\boldsymbol{u} - \Bigl(\tilde{\boldsymbol{f}} + \sum_{p=1}^{m}\sum_{q=1}^{n_p} \zeta_{pq}\boldsymbol{f}^{pq}\Bigr) = \boldsymbol{0},\ \boldsymbol{\zeta} \in \mathcal{Z}(\alpha) \right\}.
\end{aligned}
\right. \tag{21}
$$

In what follows, we may assume $\alpha > 0$ without loss of generality, because the assumption (17) and Assumption 2.1 imply $\widehat{\alpha} > 0$. Observe that $(\alpha, \boldsymbol{u})$ solves (21) if and only if

$$
\min_{\boldsymbol{\zeta}} \Bigl\{ -\Bigl\| \boldsymbol{D}\boldsymbol{K}^{-1}\Bigl(\tilde{\boldsymbol{f}} + \sum_{p=1}^{m}\sum_{q=1}^{n_p} \zeta_{pq}\boldsymbol{f}^{pq}\Bigr) \Bigr\|_{\infty} + u^{\rm c} : \boldsymbol{\zeta} \in \mathcal{Z}(\alpha) \Bigr\} \geq 0 \tag{22}
$$

is satisfied. By using the definition (19) of $\mathcal{Z}$, the left-hand side of inequality (22) is equivalent to the following series of $2n^{\rm c}$ SOCP problems in variables $\boldsymbol{\zeta} \in \mathbf{R}^{n^{\zeta}}$:

$$
\left.
\begin{aligned}
\mathcal{P}_l^{+}(\alpha, \boldsymbol{u}) : \quad & \min \quad \boldsymbol{d}_l^{\top}\boldsymbol{K}^{-1}\Bigl(\tilde{\boldsymbol{f}} + \sum_{p=1}^{m}\sum_{q=1}^{n_p} \zeta_{pq}\boldsymbol{f}^{pq}\Bigr) + u^{\rm c} \\
& \ \text{s.t.} \quad \alpha \geq \|\boldsymbol{\zeta}_p\| \quad (p = 1, \dots, m);
\end{aligned}
\right\} \tag{23}
$$

$$
\left.
\begin{aligned}
\mathcal{P}_l^{-}(\alpha, \boldsymbol{u}) : \quad & \min \quad -\boldsymbol{d}_l^{\top}\boldsymbol{K}^{-1}\Bigl(\tilde{\boldsymbol{f}} + \sum_{p=1}^{m}\sum_{q=1}^{n_p} \zeta_{pq}\boldsymbol{f}^{pq}\Bigr) + u^{\rm c} \\
& \ \text{s.t.} \quad \alpha \geq \|\boldsymbol{\zeta}_p\| \quad (p = 1, \dots, m).
\end{aligned}
\right\} \tag{24}
$$

For each $l = 1, \dots, n^{\rm c}$, the dual problem of Problem (23) is formulated in variables $(\boldsymbol{w}_l^{+}, \boldsymbol{\mu}_l^{+}) \in \mathbf{R}^{n^{\rm c}} \times \mathbf{R}^{m}$ with $\boldsymbol{w}_{lp}^{+} = (w_{lpq}^{+})_{q=1}^{n_p}$, $\boldsymbol{w}_l^{+} = (\boldsymbol{w}_{lp}^{+})_{p=1}^{m}$, and $\boldsymbol{\mu}_l^{+} = (\mu_{lp}^{+})_{p=1}^{m}$ as

$$
\left.
\begin{aligned}
\mathcal{D}_l^{+}(\alpha, \boldsymbol{u}) : \quad & \max \quad \theta_{{\rm D}_l}^{+} := \boldsymbol{d}_l^{\top}\boldsymbol{K}^{-1}\tilde{\boldsymbol{f}} + u^{\rm c} - \alpha\sum_{p=1}^{m} \mu_{lp}^{+} \\
& \ \text{s.t.} \quad \boldsymbol{d}_l^{\top}\boldsymbol{K}^{-1}\boldsymbol{f}^{pq} - w_{lpq}^{+} = 0 \quad (q = 1, \dots, n_p;\ p = 1, \dots, m), \\
& \qquad\quad \mu_{lp}^{+} \geq \|(w_{lpq}^{+})_{q=1}^{n_p}\| \quad (p = 1, \dots, m).
\end{aligned}
\right\} \tag{25}
$$

Notice here that $\alpha > 0$ guarantees the strict feasibility of $\mathcal{P}_l^{+}(\alpha, \boldsymbol{u})$, i.e., there exists a $\boldsymbol{\zeta}$ satisfying $\alpha > \|\boldsymbol{\zeta}_p\|$ $(p = 1, \dots, m)$. It is clear that $\mathcal{D}_l^{+}(\alpha, \boldsymbol{u})$ has a strictly feasible solution, because μ_{lp}^{+} is not bounded from above. Hence, by applying the conic duality theorem (Theorem 2.3) to the pair of $\mathcal{P}_l^{+}(\alpha, \boldsymbol{u})$ and $\mathcal{D}_l^{+}(\alpha, \boldsymbol{u})$, we see that the optimal objective value of $\mathcal{P}_l^{+}(\alpha, \boldsymbol{u})$ coincides with that of $\mathcal{D}_l^{+}(\alpha, \boldsymbol{u})$.

Similarly, for each $l = 1, \dots, n^{\rm c}$, the dual problem of Problem (24) is formulated in variables $(\boldsymbol{w}_l^{-}, \boldsymbol{\mu}_l^{-}) \in \mathbf{R}^{n^{\rm c}} \times \mathbf{R}^{m}$ as

$$
\left.
\begin{aligned}
\mathcal{D}_l^{-}(\alpha, \boldsymbol{u}) : \quad & \max \quad \theta_{{\rm D}_l}^{-} := -\boldsymbol{d}_l^{\top}\boldsymbol{K}^{-1}\tilde{\boldsymbol{f}} + u^{\rm c} - \alpha\sum_{p=1}^{m} \mu_{lp}^{-} \\
& \ \text{s.t.} \quad \boldsymbol{d}_l^{\top}\boldsymbol{K}^{-1}\boldsymbol{f}^{pq} - w_{lpq}^{-} = 0 \quad (q = 1, \dots, n_p;\ p = 1, \dots, m), \\
& \qquad\quad \mu_{lp}^{-} \geq \|(w_{lpq}^{-})_{q=1}^{n_p}\| \quad (p = 1, \dots, m).
\end{aligned}
\right\} \tag{26}
$$

Since both $\mathcal{P}_l^{-}(\alpha, \boldsymbol{u})$ and $\mathcal{D}_l^{-}(\alpha, \boldsymbol{u})$ are strictly feasible, Theorem 2.3 implies that the optimal objective value of $\mathcal{P}_l^{-}(\alpha, \boldsymbol{u})$ coincides with that of $\mathcal{D}_l^{-}(\alpha, \boldsymbol{u})$.

Consequently, (22) is equivalent to the fact that there exist feasible solutions $(\boldsymbol{w}_l^+, \boldsymbol{\mu}_l^+)$ and $(\boldsymbol{w}_l^-, \boldsymbol{\mu}_l^-)$ of Problems (25) and (26), respectively, satisfying $\theta_{\mathrm{D}_l}^+ \geq 0$ and $\theta_{\mathrm{D}_l}^- \geq 0$, which is reduced to

$$\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\mathrm{c}} \geq \alpha \sum_{p=1}^{m} \left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^{pq} \right)_{q=1}^{n_p} \right\| \quad (l = 1, \dots, n^{\mathrm{c}}), \tag{27}$$

$$-\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\mathrm{c}} \geq \alpha \sum_{p=1}^{m} \left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^{pq} \right)_{q=1}^{n_p} \right\| \quad (l = 1, \dots, n^{\mathrm{c}}). \tag{28}$$

It follows from (27) and (28) that Problem (16) is reduced to

$$\left. \begin{array}{ll} \max & \alpha \\ \mathrm{s.t.} & \alpha \leq \dfrac{\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\mathrm{c}}}{\sum_{p=1}^{m} \left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^{pq} \right)_{q=1}^{n_p} \right\|}, \quad \alpha \leq \dfrac{-\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\mathrm{c}}}{\sum_{p=1}^{m} \left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^{pq} \right)_{q=1}^{n_p} \right\|} \quad (l = 1, \dots, n^{\mathrm{c}}), \end{array} \right\} \tag{29}$$

which is equivalent to Problem (20). □

The proof of Proposition 5.1 is motivated by an extension of the idea found in that of Theorem 1 in [7].

It should be emphasized that the formulation (20) does not include any variable, which implies that Proposition 5.1 gives a tractable formulation for computing the robustness function $\widehat{\alpha}$. Indeed, (20) implies that $\widehat{\alpha}$ is obtained by comparing $2n^{\mathrm{c}}$ terms computed from the data. It is obvious that each term can be computed within the number of arithmetic operations bounded by a polynomial of n^{d} and n^{ζ}. Consequently, Proposition 5.1 has the advantage such that the number of arithmetic operations required is bounded by a polynomial of the dimensions of the mechanical system and the uncertainty parameters, on the contrary to the fact that the formulation (16) has infinitely many constraint conditions.

Recall that we started the discussion in Section 4 by assuming that the constraints on the mechanical performances are written as (13). Therefore, we easily see that Proposition 5.1 can be applied not only to the stress constraints but also to any constraints expressed as the linear inequalities of displacements.

We next consider a simpler case than the case of (18) and (19), where all uncertainty parameters are running through the uncertainty set uncorrelatedly under the l_2-norm constraint. Suppose that $\mathcal{T}$ and $\mathcal{Z}$ are given as

$$\mathcal{T}(\alpha, \widetilde{\boldsymbol{f}}) = \left\{ \widetilde{\boldsymbol{f}} + \sum_{q=1}^{n^\zeta} \zeta_q \boldsymbol{f}^q \,\middle|\, \boldsymbol{\zeta} \in \mathcal{Z}(\alpha) \right\}, \tag{30}$$

$$\mathcal{Z}(\alpha) = \{ \boldsymbol{\zeta} \in \mathbf{R}^{n^\zeta} \mid \alpha \geq \|\boldsymbol{\zeta}\|_2 \}. \tag{31}$$

In this case, the robustness function $\widehat{\alpha}$ can be obtained by putting $m = 1$ in Proposition 5.1, i.e.,

$$\widehat{\alpha}(\boldsymbol{a}, u^{\mathrm{c}}) = \min_{l=1,\dots,n^{\mathrm{c}}} \left\{ \min \left\{ \frac{\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\mathrm{c}}}{\left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^{1q} \right)_{q=1}^{n_1} \right\|_2}, \frac{-\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\mathrm{c}}}{\left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^{1q} \right)_{q=1}^{n_1} \right\|_2} \right\} \right\}. \tag{32}$$

Recall that the l_∞-norm and l_1-norm of $\boldsymbol{p} \in \mathbf{R}^n$ are defined by (1) and (2), respectively. The result for the l_∞-norm constrained uncertainty is summarized as the following proposition:

Proposition 5.2. *Assume that $\mathcal{T}$ and $\mathcal{Z}$ are given as*

$$\mathcal{T}(\alpha, \widetilde{\boldsymbol{f}}) = \Big\{ \widetilde{\boldsymbol{f}} + \sum_{p=1}^{n^\zeta} \zeta_p \boldsymbol{f}^p \Big| \boldsymbol{\zeta} \in \mathcal{Z}(\alpha) \Big\}, \tag{33}$$

$$\mathcal{Z}(\alpha) = \{ \boldsymbol{\zeta} \in \mathbf{R}^{n^\zeta} | \alpha \geq \|\boldsymbol{\zeta}\|_\infty \}. \tag{34}$$

Then the robustness function $\widehat{\alpha}$ is explicitly obtained as

$$\widehat{\alpha}(\boldsymbol{a}, u^{\rm c}) = \min_{l=1,\dots,n^{\rm c}} \left\{ \min \left\{ \frac{\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\rm c}}{\left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^p \right)_{p=1}^{n^\zeta} \right\|_1}, \frac{-\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \widetilde{\boldsymbol{f}} + u^{\rm c}}{\left\| \left(\boldsymbol{d}_l^\top \boldsymbol{K}^{-1} \boldsymbol{f}^p \right)_{p=1}^{n^\zeta} \right\|_1} \right\} \right\}. \tag{35}$$

Proof. Proposition 5.2 is obtained as a particular case of Proposition 5.1 by putting $n_p = 1$ ($p = 1, \dots, m$). □

It is clear that the definition (34) of $\mathcal{Z}(\alpha)$ implies that $\zeta_1, \dots, \zeta_{n^\zeta}$ perturb independently from each other. The box-constrained uncertainty, which is often used in the interval analysis [10], is included in Proposition 5.2 as a special case of

$$f_j^p = \begin{cases} 1 & (j = p) \\ 0 & (\text{otherwise}) \end{cases} \qquad (p = 1, \dots, n^\zeta).$$

6. Numerical experiments.

The robustness functions are computed for various trusses by using the methods proposed in Section 5. Computation has been carried out on Pentium M (1.5GHz with 1GB memory) with MATLAB Version 6.5.1 [21].

6.1. 2-bar truss.

Consider a two-bar truss illustrated in Fig.2. The nodal coordinates at the initial unstressed state of nodes (a), (b), and (c) are specified as $(x, y) = (1, 1)$, $(0, 1)$, and $(0, 0)$, respectively. Nodes (b) and (c) are pin-supported; i.e., $n^{\rm d} = 2$ and $n^{\rm m} = 2$. The cross-sectional areas $\boldsymbol{a}$ and the nominal external forces $\widetilde{\boldsymbol{f}}$ are given by

$$\boldsymbol{a} = (20, 40)^\top, \quad \widetilde{\boldsymbol{f}} = (10, 0)^\top.$$

Consider the stress constraints of all members defined by (14), where

$$E = 10.0, \quad \sigma_i^{\rm c} = 1.0 \quad (i = 1, 2).$$

However, the robustness function does not depend on E, because only the stress constraints are considered.

Suppose that the uncertain set $\mathcal{T}$ is defined by (18) and (19) in Proposition 5.1, where $n_1 = 2$ and $m = 1$ with

$$\boldsymbol{f}^{11} = (1/\sqrt{2}, 1/\sqrt{2}), \quad \boldsymbol{f}^{12} = (1/\sqrt{2}, -1/\sqrt{2}). \tag{36}$$

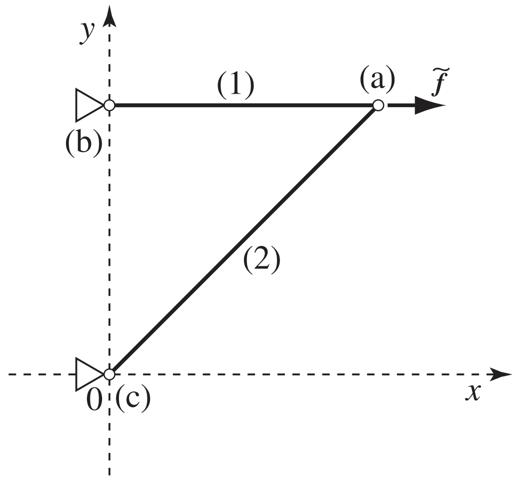

Figure 2: 2-bar truss.

The corresponding admissible set of the external force vector $\boldsymbol{f}$ is as illustrated in Fig.3, i.e., the vector $\boldsymbol{f} = \widetilde{\boldsymbol{f}} + \sum_{q=1}^{2} \zeta_{1q} \boldsymbol{f}^{1q}$ should exist in the interior or on the boundary of the circle centered at $\widetilde{\boldsymbol{f}}$ with the radius of α. By computing (20), we obtain $\widehat{\alpha}(\boldsymbol{a}, \sigma^{\mathrm{c}}) = 7.0711$. In order to verify this result, we randomly generate a number of $\boldsymbol{\zeta} \in \mathcal{Z}(\widehat{\alpha})$, and compute the corresponding member stresses σ_i. The points obtained on the $\sigma_1\sigma_2$-plane are shown in Fig.5. It is observed from Fig.5 that the stress constraints (14) are satisfied with all possible $\boldsymbol{\zeta}$. The worst case of this example corresponds to $\boldsymbol{\zeta} = (5.0, -5.0)^\top$, where the constraint $\sigma_1 \leq \sigma_1^{\mathrm{c}}$ becomes active.

We next take

$$\boldsymbol{f}^{11} = (1/\sqrt{2}, 1/\sqrt{2}), \quad \boldsymbol{f}^{12} = (0.5, -0.5), \tag{37}$$

then we obtain $\widehat{\alpha}(\widetilde{\boldsymbol{a}}, \sigma^{\mathrm{c}}) = 10.0$ by computing (20). The worst case of this example corresponds to $\boldsymbol{\zeta} = (0, 10)^\top$, where the constraint $\sigma_1 \leq \sigma_1^{\mathrm{c}}$ becomes active.

Consider the l_∞-norm constrained uncertain set $\mathcal{T}$ defined by (33) and (34) in Proposition 5.2 with (37) and $n^\zeta = 2$. The corresponding admissible set of $\boldsymbol{f}$ is as illustrated in Fig.4. Then, by using (35), the robustness function is directly obtained as $\widehat{\alpha}(\boldsymbol{a}, \sigma^{\mathrm{c}}) = 12.4264$. Fig.6 shows (σ_1, σ_2) corresponding to the randomly generated $\boldsymbol{\zeta} \in \mathcal{Z}(\widehat{\alpha}(\boldsymbol{a}, \sigma^{\mathrm{c}}))$. In this example, it is observed from Fig.6 that the constraints $\sigma_2 \leq \sigma_2^{\mathrm{c}}$ and $\sigma_2 \geq -\sigma_2^{\mathrm{c}}$ possibly become active in the worst cases. The corresponding uncertain parameters are $\boldsymbol{\zeta} = (12.4264, -12.4264)^\top$ and $\boldsymbol{\zeta} = (-12.4264, 12.4264)^\top$, respectively. It is interesting to see that, not only the values of robustness function, but also the critical constraints at the worst cases depend on the definition of uncertainty sets.

6.2. 29-bar truss.

Consider a truss illustrated in Fig.7. Nodes (a) and (b) are pin-supported at $(x, y) = (0, 100)$ and $(0, 0)$, respectively, where $n^{\mathrm{d}} = 20$ and $n^{\mathrm{m}} = 29$. The lengths of members both in x- and y-directions are 50.0. Two loads $(0, -1)$ are applied at nodes (c) and (d) as the nominal external loads $\widetilde{\boldsymbol{f}}$. Consider the stress constraints of all members defined by (14), where

$$E = 100.0, \quad \sigma_i^{\mathrm{c}} = 5.0 \quad (i = 1, 2).$$

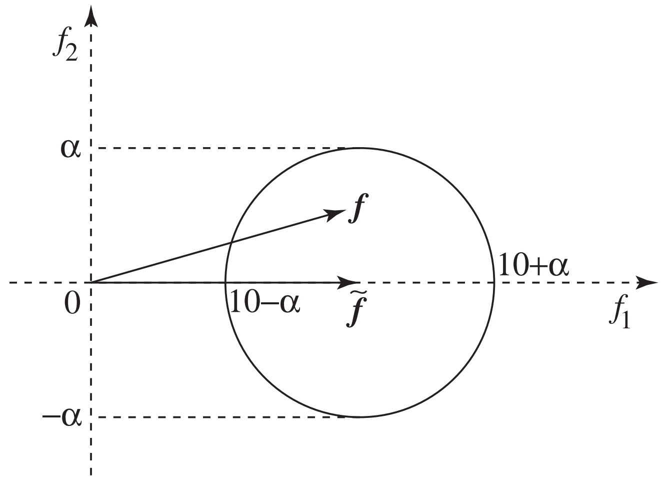

Figure 3: Admissible set of $\boldsymbol{f}$ defined by (18), (19), and (36).

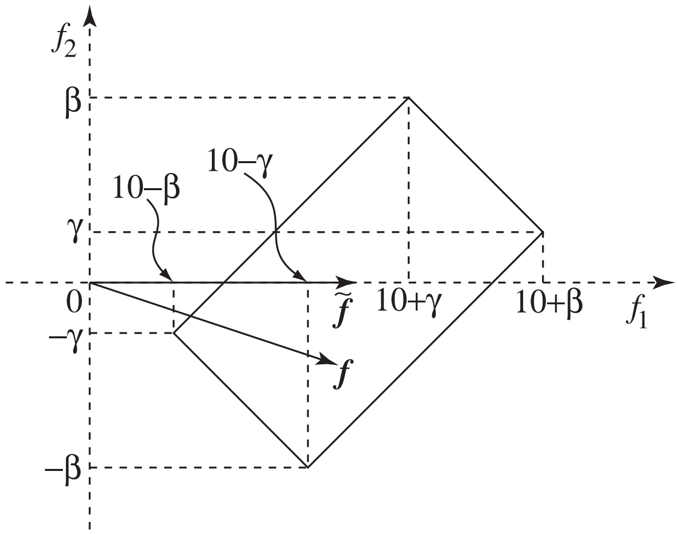

Figure 4: Admissible set of $\boldsymbol{f}$ defined by (33), (34), and (37), where $\beta = \alpha(\sqrt{2}+1)/2$ and $\gamma = \alpha(\sqrt{2}-1)/2$.

Suppose that the uncertain set $\mathcal{T}$ is defined by (18) and (19) in Proposition 5.1, where $n_1 = n^{\mathrm{d}}$ and $m = 1$. Let

$$f_j^{1q} = \begin{cases} 1 & (j = q) \\ 0 & (\text{otherwise}) \end{cases} \quad (q = 1, \dots, n^{\mathrm{d}}), \tag{38}$$

$$a_i^1 = 40.0 \quad (i = 1, \dots, n^{\mathrm{m}}). \tag{39}$$

Hence, there may exist uncertain external loads at all the unconstrained nodes. By computing (20), we obtain $\widehat{\alpha}(\boldsymbol{a}^1, \sigma^{\mathrm{c}}) = 0.78477$. As an alternative set of cross-sectional areas, consider

$$a_i^2 = \begin{cases} 15 & (i = 9, 15, 18, 27), \\ 65 & (i = 1, 2, 24, 26), \\ 40 & (\text{otherwise}). \end{cases} \tag{40}$$

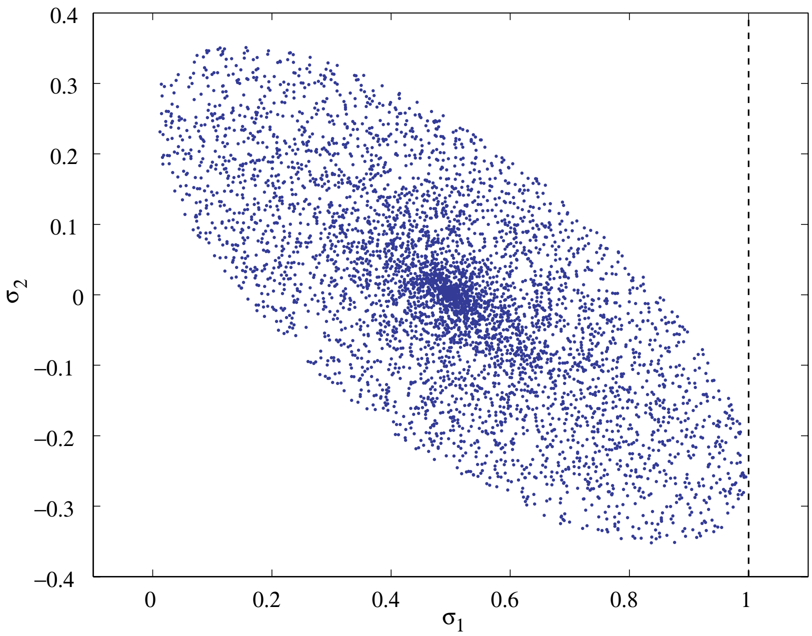

Figure 5: Stress states of the 2-bar truss for randomly generated $\boldsymbol{\zeta}$ satisfying (18), (19), and (36) with $\alpha = \widehat{\alpha}$.

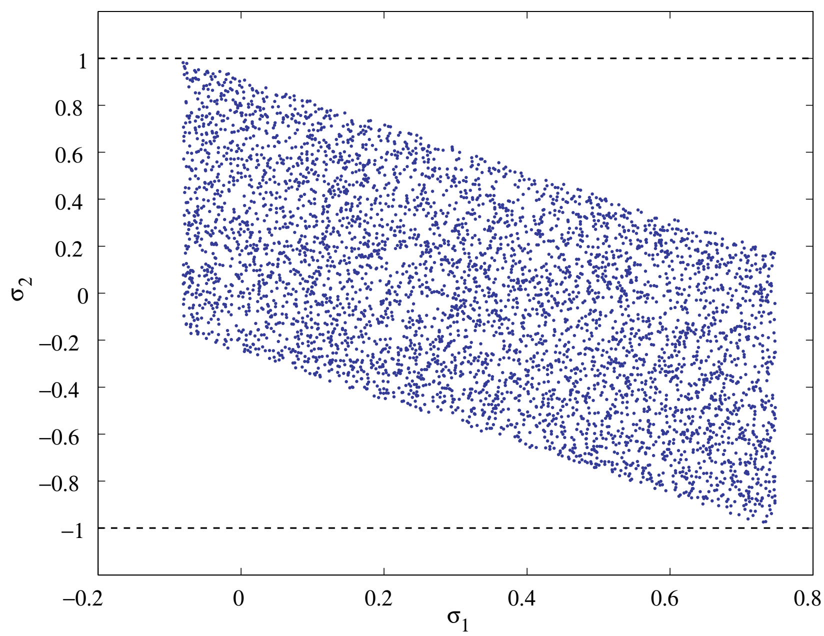

Figure 6: Stress states of the 2-bar truss for randomly generated $\boldsymbol{\zeta}$ satisfying (33), (34), and (37) with $\alpha = \widehat{\alpha}$.

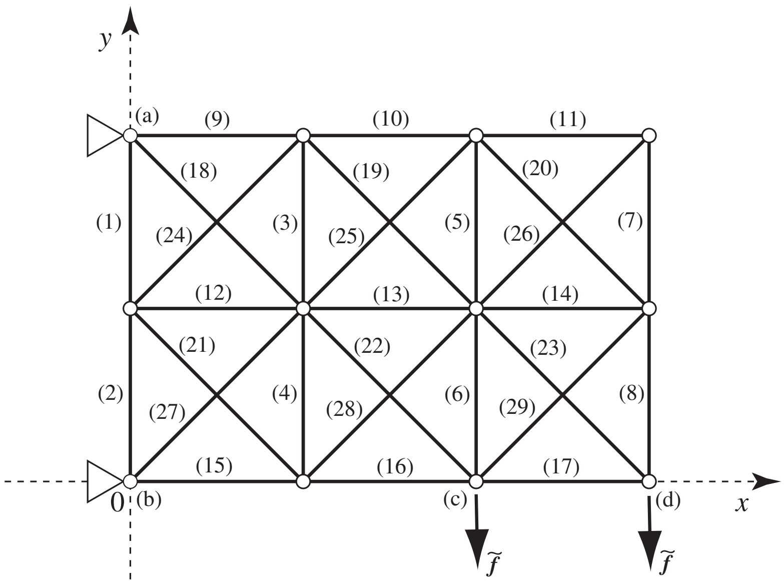

Figure 7: 29-bar truss.

As a solution of (20), we obtain $\widehat{\alpha}(\boldsymbol{a}^2, \sigma^{\rm c}) = 1.6778$. Hence, the robustness function of $\boldsymbol{a}^2$ is more than twice larger than that of $\boldsymbol{a}^1$, despite the fact that $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$ have the same structural volume.

In order to verify this result, we randomly generate a number of $\boldsymbol{\zeta} \in \mathcal{Z}(\widehat{\alpha})$, and compute the corresponding member stresses σ_i. Fig.8 and Fig.9 show the obtained σ_i from the randomly generated $\boldsymbol{\zeta} \in \mathcal{Z}(\alpha)$ with $\alpha = \widehat{\alpha}(\boldsymbol{a}^1, \sigma^{\rm c})$ and $\alpha = \widehat{\alpha}(\boldsymbol{a}^2, \sigma^{\rm c})$, respectively. It is observed from Fig.8 and Fig.9 that the stress constraints (14) are satisfied with any possible $\boldsymbol{\zeta}$. Recall that our formulations in Section 5 provide the exact value of $\widehat{\alpha}$ in the sense that there exists a worst-case uncertainty parameter $\overline{\boldsymbol{\zeta}} \in \mathcal{Z}(\widehat{\alpha})$ at which some stress constraints become active. However, the results in Fig.8 and Fig.9 seem to be conservative. This is because the actual worst-case behavior cannot be accurately predicted, in general, by taking a rather small number of random samples.

For $\boldsymbol{a} = \boldsymbol{a}^1$, it is conjectured from Fig.8 that the worst case of this example corresponding to $\boldsymbol{\zeta}$ such that $\sigma_9 \leq \sigma_9^{\rm c}$ and/or $\sigma_{15} \geq -\sigma_{15}^{\rm c}$ become active. Hence, it is natural to expect that the robustness function increases if we increase a_9 and a_{15}, which has been confirmed by showing $\widehat{\alpha}(\boldsymbol{a}^1, \sigma^{\rm c}) < \widehat{\alpha}(\boldsymbol{a}^2, \sigma^{\rm c})$.

7. Conclusions.

In this paper, we have proposed a direct method for calculating the robustness function associated with stress constraints, which may permit us to apply the info-gap decision theory [2] to the structural reliability design.

We have shown that the robustness function under uncertain external forces can be obtained as the optimal objective value of an optimization problem with finite number of variables and infinitely many constraint conditions. Particularly, we have investigated the uncertainty sets which are expressed via some Euclidean norms of a vector of uncertainty parameters. By using the strong duality theory of the second-order cone programming problem, we have reformulated the present problem to a numerically tractable form without any variable.

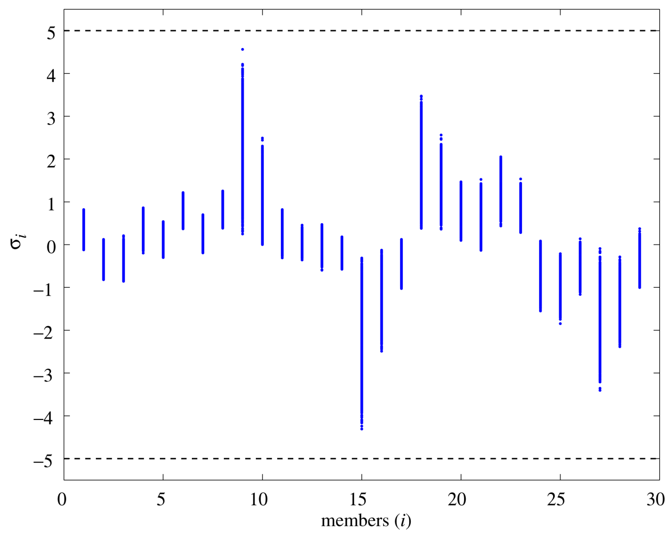

Figure 8: Stress states of the 29-bar truss with $\boldsymbol{a} = \boldsymbol{a}^1$ for randomly generated $\boldsymbol{\zeta}$ with $\alpha = 0.78477$.

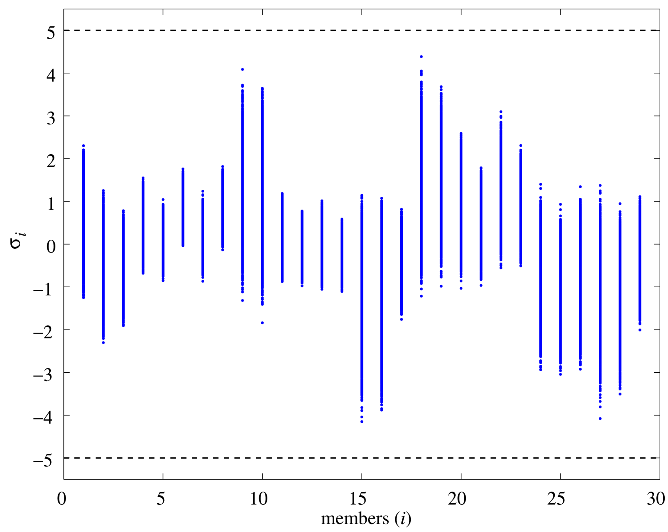

Figure 9: Stress states of the 29-bar truss with $\boldsymbol{a} = \boldsymbol{a}^2$ for randomly generated $\boldsymbol{\zeta}$ with $\alpha = 1.6778$.

The number of arithmetic operations required to compute the robustness function in the proposed formulation is bounded by a polynomial of the dimensions of mechanical system and uncertainty parameters. This indicates its excellent performance even in large scale problems. It is straightforward to show that the present formulation for stress constraints can be extended to the cases including linear inequality constraints of the displacements.

The robustness functions of trusses have been obtained under various conditions of uncertainties in the numerical examples. It has been shown that the robustness function depends on the member cross-sectional areas as well as the definition of the uncertainty sets. We have also illustrated that the robustness function can be increased by stiffening the members on which the stress constraints become active in the worst case. This intuitive scheme of structural design may suggest a novel and promising concept of the robust structural optimization based on the robustness function, which remains as our future work.

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