Bridging reliability-based design optimization and robust design optimization: A data-driven approach

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RBDO & RO

- RBDO (= reliability-based design optimization)
  - probabilistic model
    - asm. of an input distribution
    - cstr. on structural reliability
- RO (= robust optimization)
  - possibilistic model
    - asm. of an uncertainty set
    - cstr. in worst-case scenario

RBDO & RO

- RBDO (= reliability-based design optimization)
  - a little *more difficult*
- RO (= robust optimization)
  - a little *easier*

    (altho. already *more difficult* than usual optim.)
    - tractable (sometimes *convex*) reformulations

      [Elishakoff, Haftka & Fang '94], [Pantelides & Ganzerli '98]
      [Ben-Tal & Nemirovski '97], [Cherkaev & Cherkaev '03, '08]
      [K. & Takewaki '06], [de Gournay, Allaire & Jouve '08], [K. & Guo '10]
      [Yonekura & K. '10], [Takezawa, Nii, Kitamura & Kogiso '11]
      [Holmberg, Thore & Klarbring '15], etc.
    - conservative approximations [Guo, Bai, Zhang & Gao '09]
      [Guo, Du & Gao '11], [Hashimoto & K. '15],

RBDO w/ uncertainty

- asm.: statistical information is *incomplete*.
- structural reliability $r(\boldsymbol{x})$
- target reliability constraint

$$r(\boldsymbol{x}) \geq 1 - \epsilon$$

[Gunawan & Papalambros ’06], [Youn & Wang ’08]
[Noh, Choi, Lee, Gorsich & Lamb ’11], [Zaman & Mahadevan ’17]
[Moon, Cho, Choi, Gaul, Lamb & Gorsich ’17, ’18], [Ito, Kim & Kogiso ’18]

RBDO w/ uncertainty

- asm.: statistical information is *incomplete*.
- structural reliability $r(\boldsymbol{x})$ → a random variable
  (∵ input distribution is stochastic)
- target reliability constraint

$$r(\boldsymbol{x}) \geq 1 - \epsilon$$

→ is meaningful only in a probabilistic manner:

[Gunawan & Papalambros '06], [Youn & Wang '08]
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RBDO w/ uncertainty

- asm.: statistical information is *incomplete*.
- structural reliability $r(\boldsymbol{x})$ $\rightarrow$ a random variable
  ($\because$ input distribution is stochastic)
- target reliability constraint

$$r(\boldsymbol{x}) \geq 1 - \epsilon$$

$\rightarrow$ is meaningful only in a probabilistic manner:

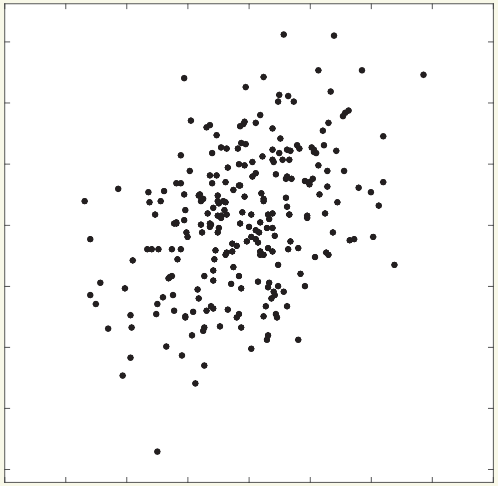
$$\mathrm{P}\{r(\boldsymbol{x}) \geq 1 - \epsilon\} \geq 1 - \delta \ .$$

- $1 - \delta$: target confidence level
- We say that, e.g., it is 90% sure that $\boldsymbol{x}$ has 93% reliability.
  $(\delta = 0.1,\ \epsilon = 0.07)$

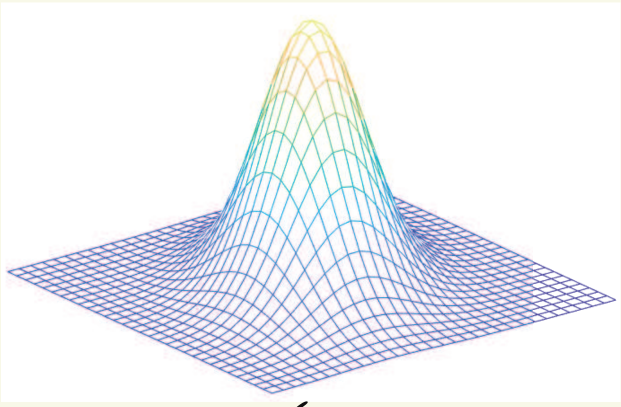
[Gunawan & Papalambros '06], [Youn & Wang '08]
[Noh, Choi, Lee, Gorsich & Lamb '11], [Zaman & Mahadevan '17]
[Moon, Cho, Choi, Gaul, Lamb & Gorsich '17, '18], [Ito, Kim & Kogiso '18]

An idea is to *construct*...

- Given: a set of random samples
  - We do *not* estimate the (cumulative) distribution function F.
- RO that conservatively approximates RBDO w/ uncertainty.
  - many *powerful methods* in RO are available
  - construct an uncertainty set from the data
  - free from error in statistical modeling process



given data



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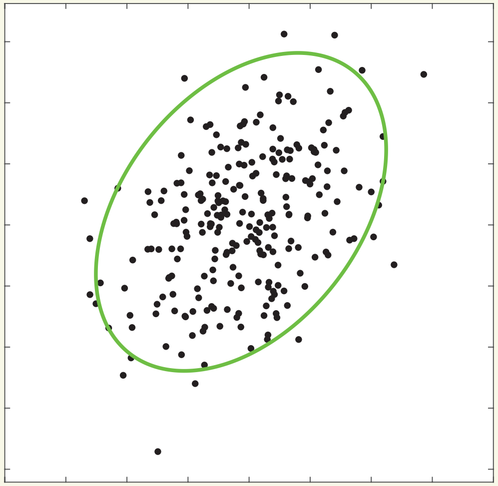

uncertainty set

given: a set of random samples...

- Derive: a sufficient condition
  - that guarantees

$$\mathrm{P}_F\Bigl\{\underbrace{\mathrm{P}_{\boldsymbol{q}}\{g_l(\boldsymbol{x};\boldsymbol{q}) \le 0\} \ge 1-\epsilon}_{\text{reliability constraint}}\Bigr\} \ge 1-\delta \ .$$

  - that has the form

$$g_l(\boldsymbol{x};\boldsymbol{q}) \le 0, \quad \forall \boldsymbol{q} \in Q \qquad \text{(robust cstr.)}$$

w/ ellipsoidal set Q.

given: a set of random samples...

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  - that has the form

$$g_l(\boldsymbol{x};\boldsymbol{q}) \le 0, \quad \forall \boldsymbol{q} \in Q \qquad \text{(robust cstr.)}$$

w/ ellipsoidal set Q.

- ...is realized with "sufficiently large Q" s. t.

$$\mathrm{P}_F \left\{ \mathrm{P}_{\boldsymbol{q}}\{\boldsymbol{q} \in Q\} \ge 1-\epsilon \right\} \ge 1-\delta .$$

  - for finding such an ellipsoid Q,
    an idea in [Hong, Huang & Lam '17] is *very useful*.

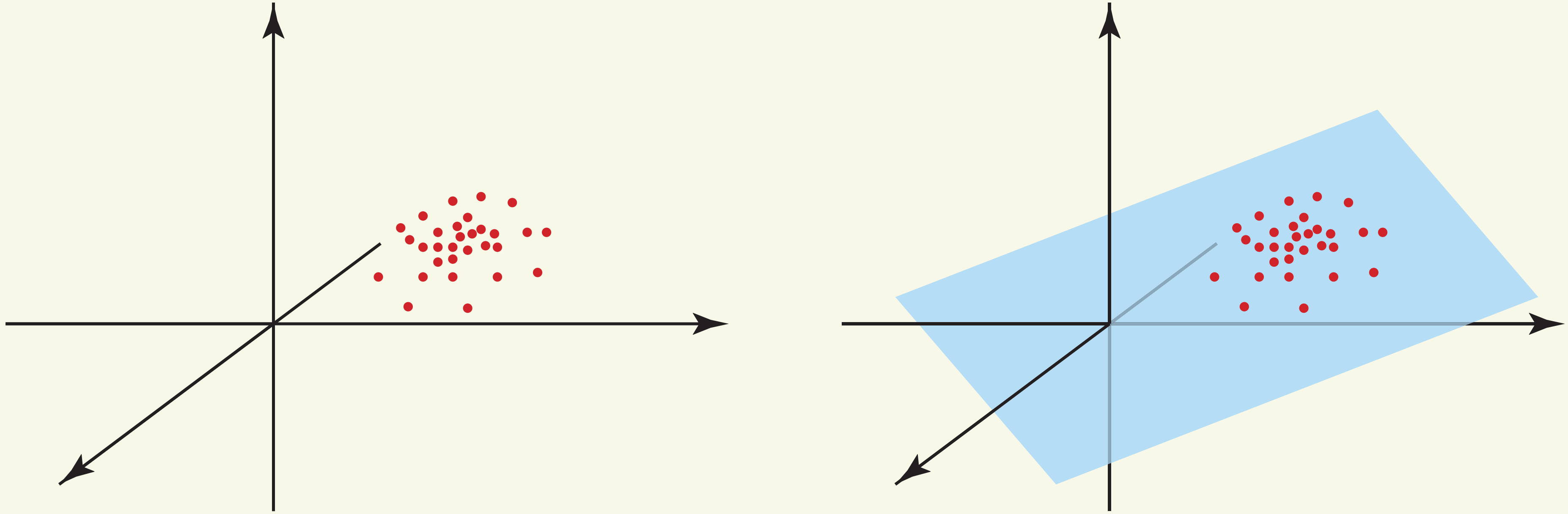
how to obtain uncertainty set Q

- 3 steps [Hong, Huang & Lam ’17]

1. dimensionality reduction
2. shape determination
3. size calibration

how to obtain uncertainty set Q

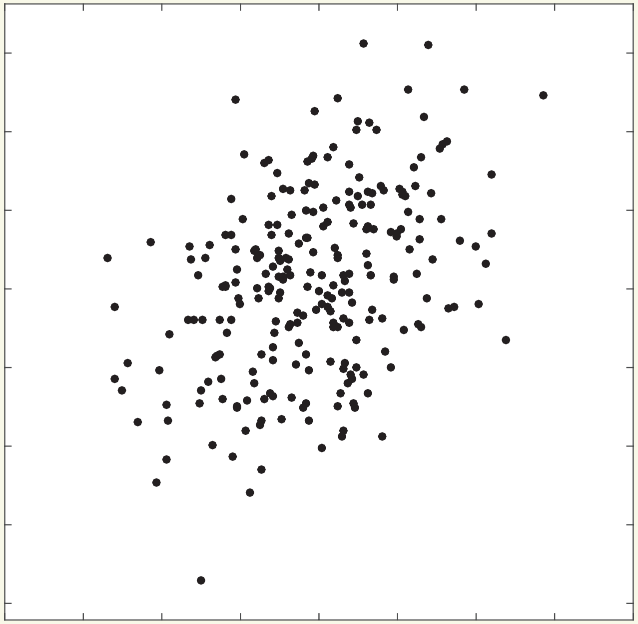
- Apply PCA (principal component analysis) to the samples,
  - for avoiding that the method becomes overly conservative.
  - $\because$ a high-dimensional ellipsoid (containing sample points) has huge volume (= curse of dimensionality).



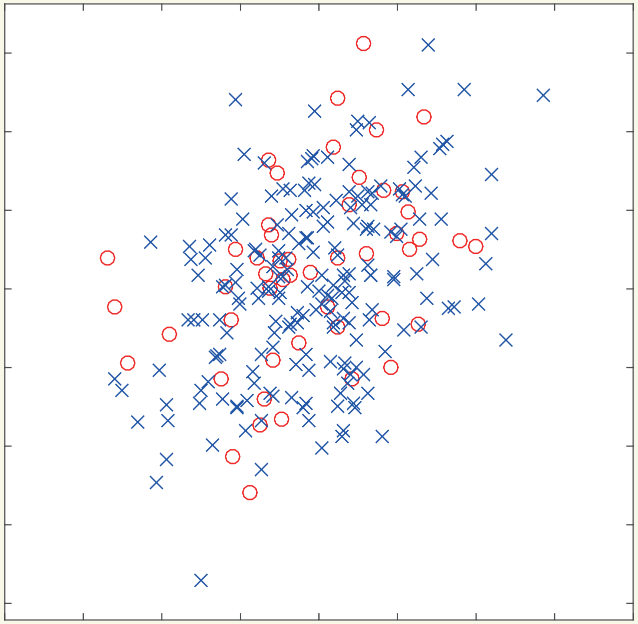
1. dimensionality reduction
2. shape determination
3. size calibration

how to obtain uncertainty set Q

- After dimensionality reduction,
- divide the set into two subsets randomly.
  - To be used in steps 2 & 3, resp.



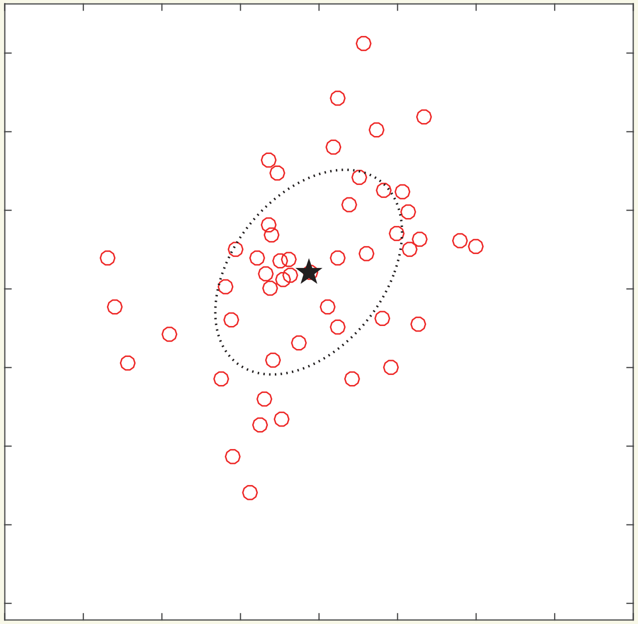
→



1. dimensionality reduction
2. shape determination
3. size calibration

how to obtain uncertainty set Q

- learn ellipsoidal shape from the samples, e.g.:
  - center $\leftarrow$ mean of the samples
  - principal axes & radii $\leftarrow$ covariance matrix of the samples



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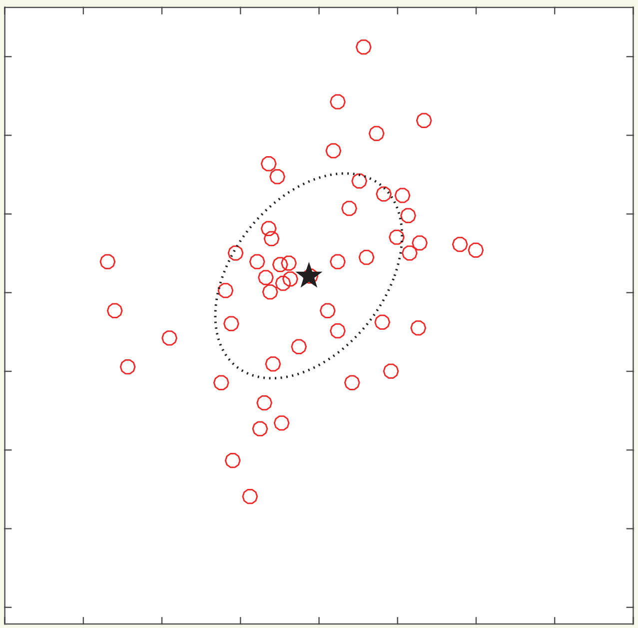


Remark:
Any shape (other than ellipsoid) can be adopted.

- box
- parallelogram
- nonconvex set, etc.

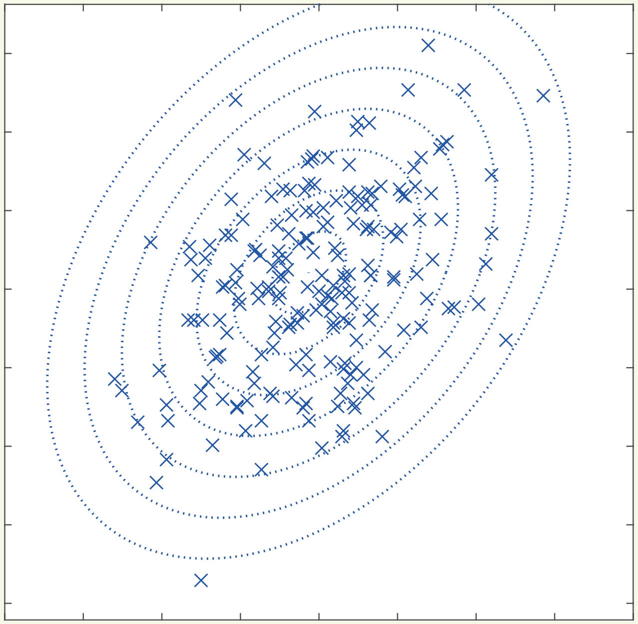
1. dimensionality reduction
2. shape determination
3. size calibration

how to obtain uncertainty set Q

- Q should satisfy

$$\mathrm{P}_F\left\{\mathrm{P}_{\boldsymbol{q}}\{\boldsymbol{q} \in Q\} \geq 1-\epsilon\right\} \geq 1-\delta \ .$$

- a sufficient condition provided by order statistics is:



- n := #samples.
- $\tilde{p}$:= minimum natural number p s. t.

$$\sum_{k=p}^{n} \binom{n}{k} (1-\epsilon)^k \epsilon^{n-k} \leq \delta \ .$$

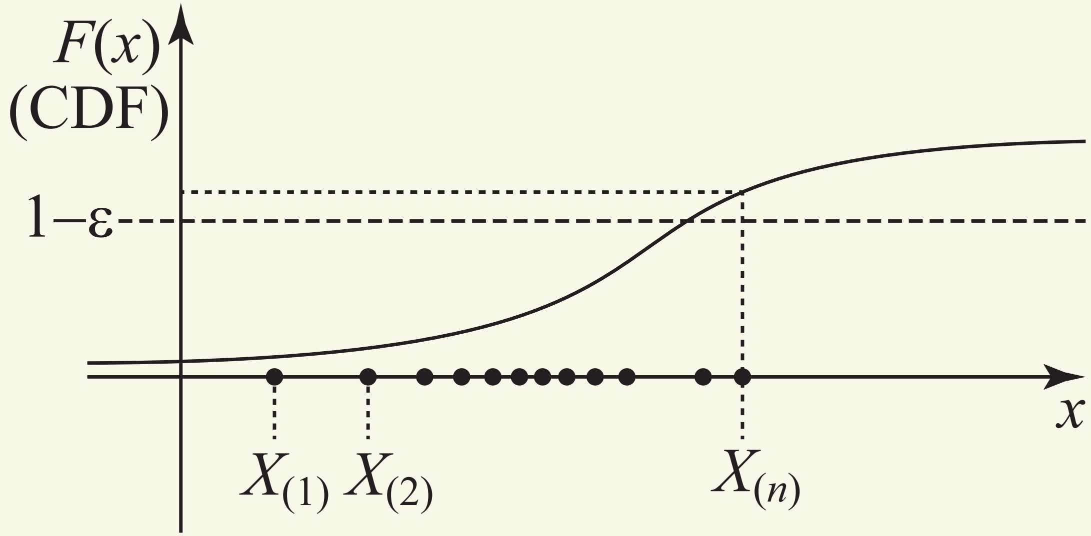
- Adjust size of Q s. t.
  Q contains $\tilde{p}$ samples.

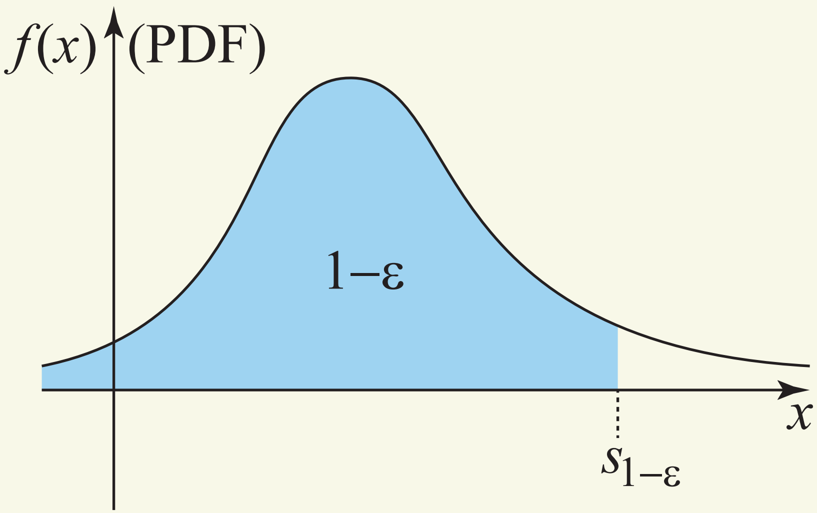
1. dimensionality reduction
2. shape determination [Hong, Huang & Lam '17]
3. size calibration: gives guarantee of confidence level

order statistics: essentials

- random samples $X_1, X_2, \ldots, X_n$
- order statistics $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ (re-numbering)
- If, e.g., "uncertainty set" covers all data points, then

$$\begin{aligned} \mathrm{P}\{F(X_{(n)}) > 1-\epsilon\} &= \mathrm{P}\{X_{(n)} > s_{1-\epsilon}\} \\ &= 1 - \mathrm{P}\{X_{(n)} \le s_{1-\epsilon}\} \end{aligned}$$





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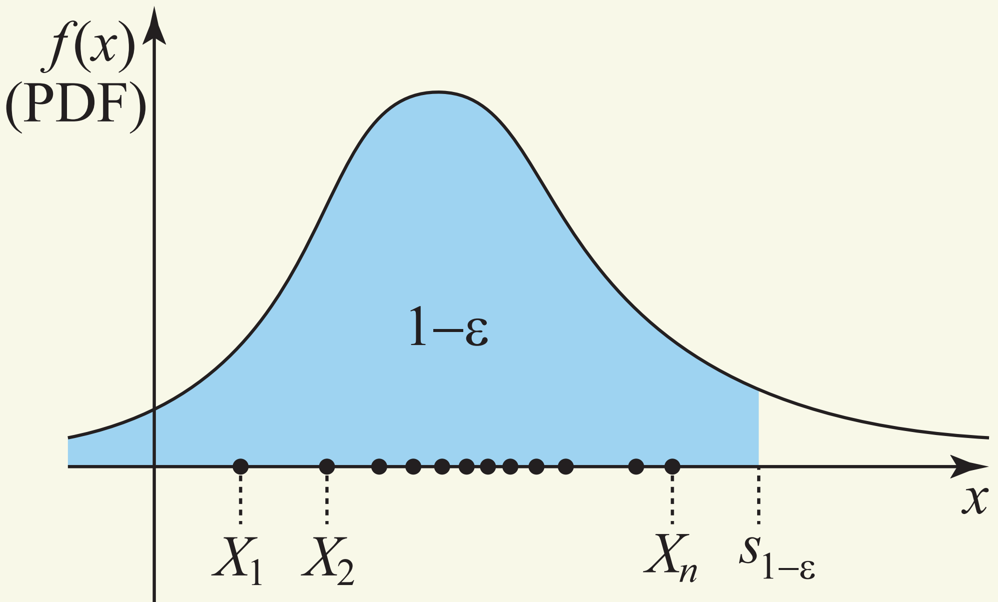
&= 1 - \mathrm{P}\{X_{(n)} \le s_{1-\epsilon}\} \\
&= 1 - \mathrm{P}\{X_1 \le s_{1-\epsilon}\} \times \cdots \times \mathrm{P}\{X_n \le s_{1-\epsilon}\} \\
&= 1 - (1-\epsilon)^n = 1 - \delta\,.
\end{aligned}
$$



order statistics: essentials

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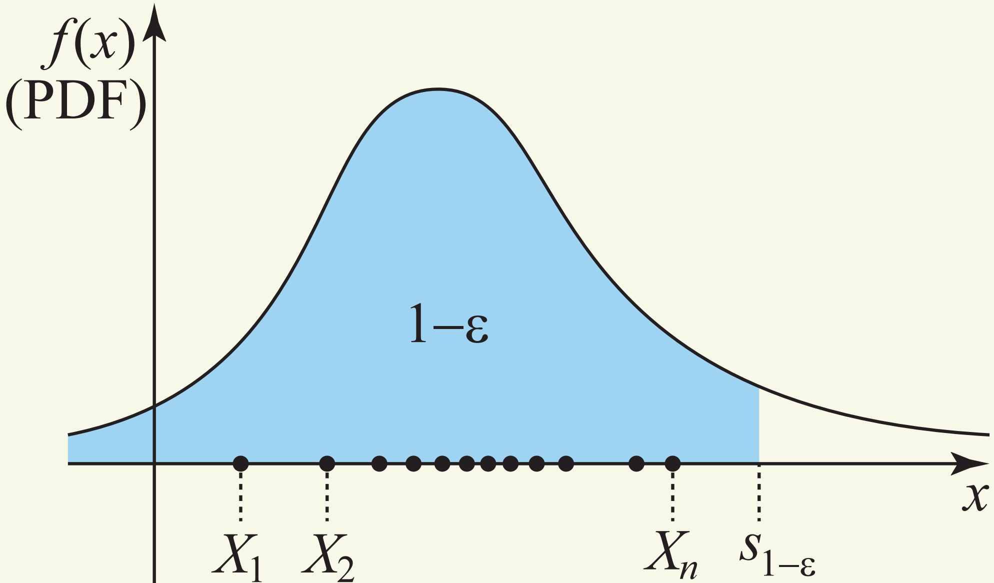


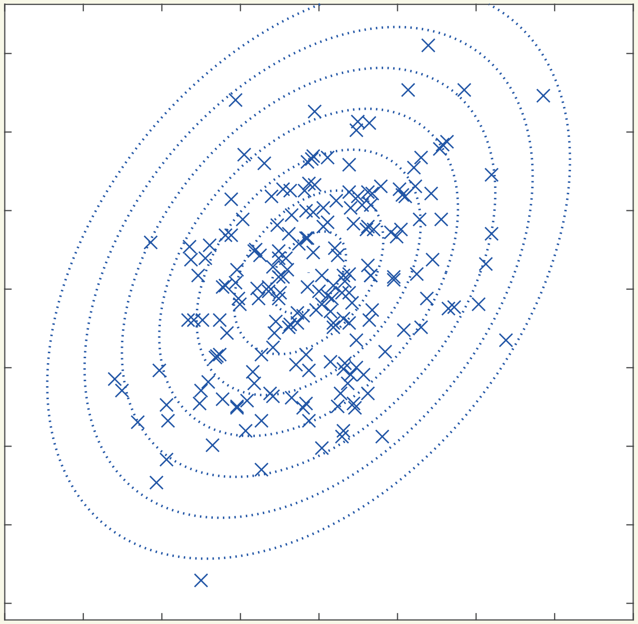
- e.g., $n = 200$, then
  - $\epsilon = 0.02 \Rightarrow \delta = 0.0176$
    98% reliability
    w/ 98.2% conf.
  - $\epsilon = 0.01 \Rightarrow \delta = 0.134$
    99% reliability
    w/ 86.6% conf.

how to obtain uncertainty set Q

- Q should satisfy

$$\mathrm{P}_F\left\{\mathrm{P}_{\boldsymbol{q}}\{\boldsymbol{q} \in Q\} \geq 1 - \epsilon\right\} \geq 1 - \delta \ .$$

- a sufficient condition provided by order statistics is:



- $n :=$ #samples.
- $\tilde{p} :=$ minimum natural number p s. t.

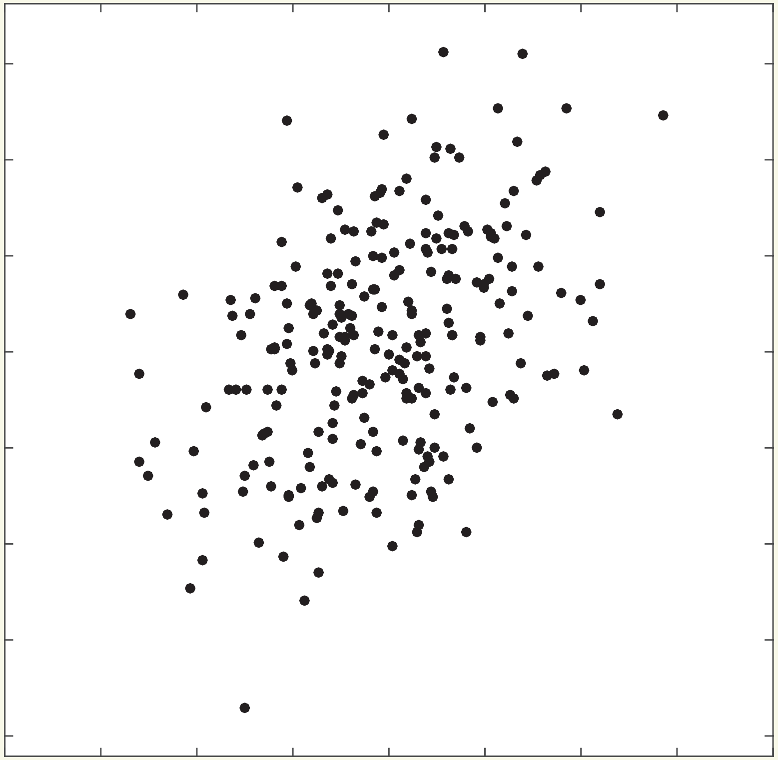
$$\sum_{k=p}^{n} \binom{n}{k} (1-\epsilon)^k \epsilon^{n-k} \leq \delta \ .$$

- Adjust size of Q s. t. Q contains $\tilde{p}$ samples.

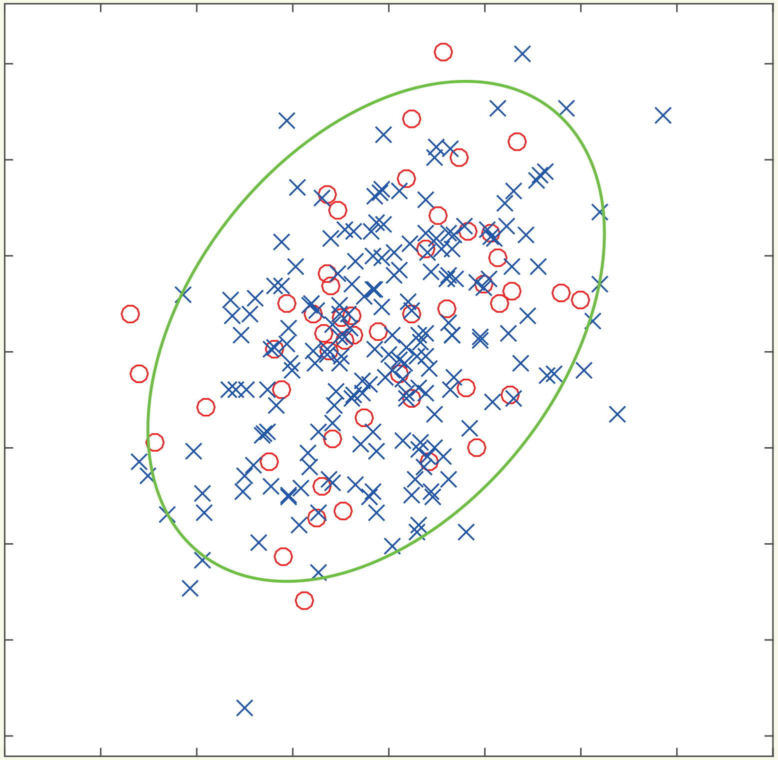
1. dimensionality reduction
2. shape determination [Hong, Huang & Lam '17]
3. size calibration: gives guarantee of confidence level

how to obtain uncertainty set Q

- We thus have obtained Q (by using *only the given data*)!



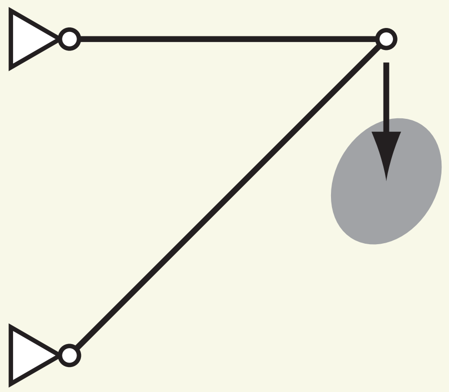
$\longrightarrow$

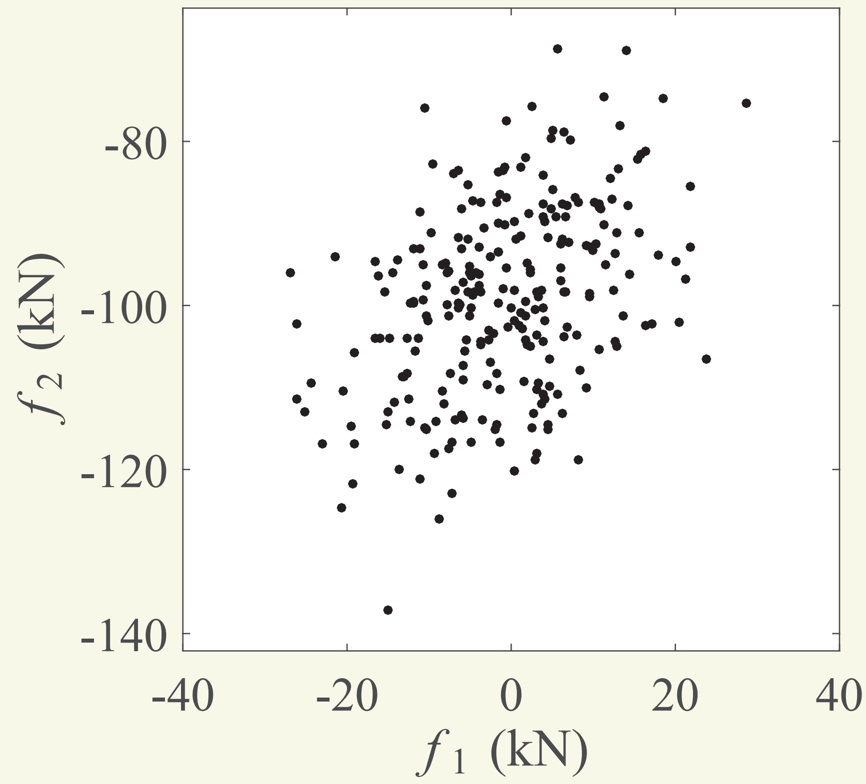


- $\hookrightarrow$ perform "robust optim. w/ this Q"

1. dimensionality reduction
2. shape determination
3. size calibration

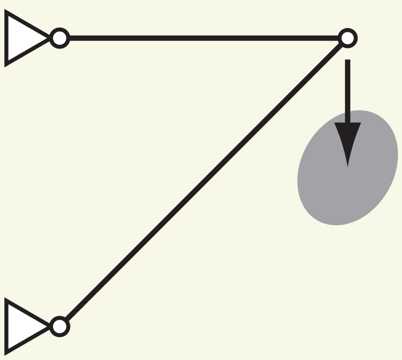
numerical example: 1) 2-bar truss (stress cstr.)

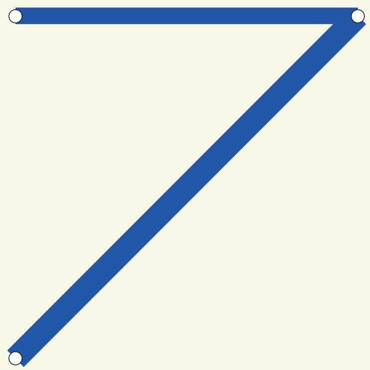


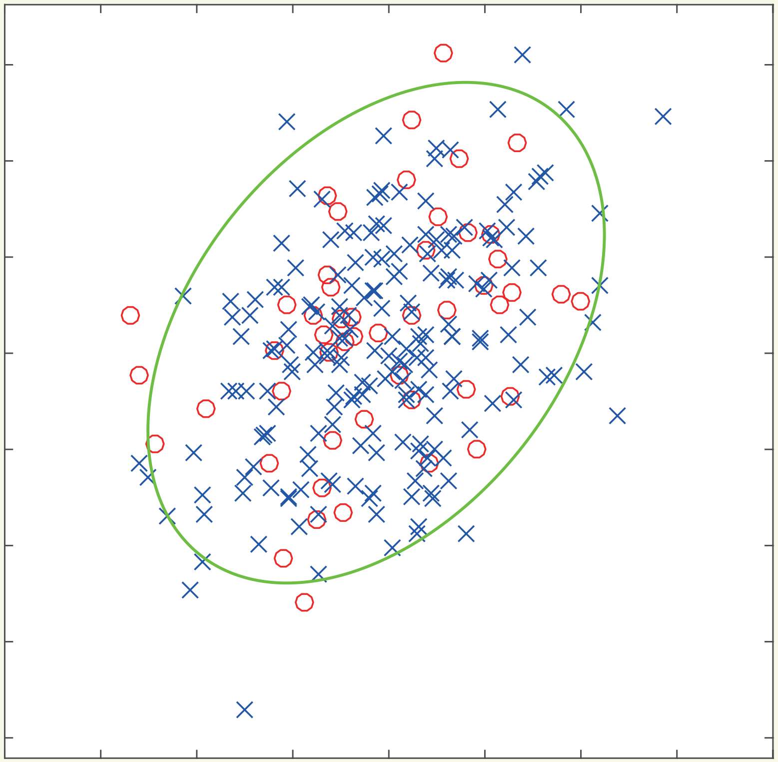


- given: 250 random samples of external loads
- volume min.
- stress cstr.: $|\sigma_i| \le \sigma_c$
  - reliability counterpart: $\mathrm{P}\{|\sigma_i| \le \sigma_c\} \ge 1 - \epsilon$ w/ confidence $1 - \delta$

ex.) 2-bar truss (stress cstr.)





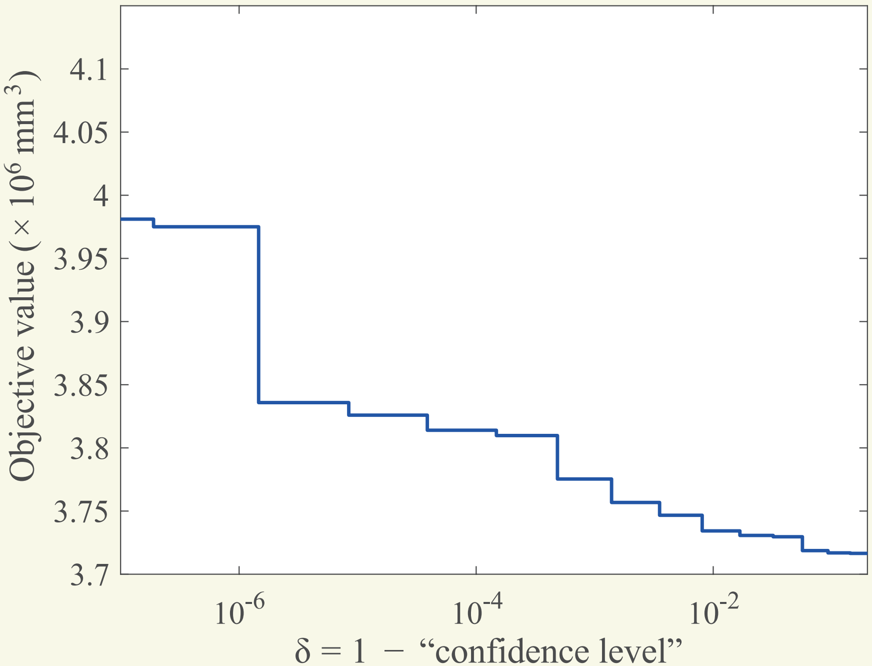


- 200 data points used for size calibration.
  - "reliability $1 - \epsilon = 0.9$" & "confidence $1 - \delta = 0.92$"

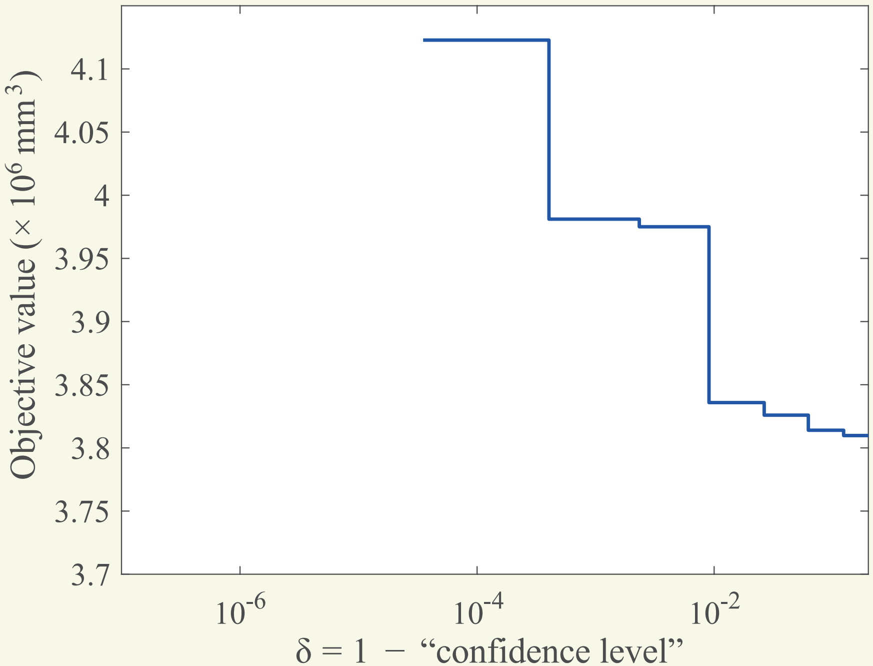
    $\Uparrow$
  - Ellipsoid Q contains 187 data points.

ex.) 2-bar truss (stress cstr.)

- guarantee $1 - \epsilon$ reliability, with $1 - \delta$ confidence
  - trade-off btw. δ & optimal value, w/ fixed $1 - \epsilon$



$1 - \epsilon = 0.90$

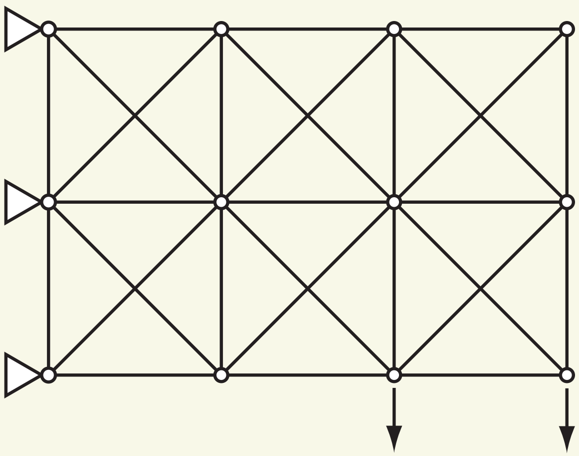


$1 - \epsilon = 0.95$

- Order statistics limits $1 - \delta \leq 1 - (1 - \epsilon)^n$.
  - $1 - \delta \leq 1 - (3.5 \times 10^{-5})$ for $(1 - \epsilon, n) = (0.95, 200)$
    - To increase confidence level, we need more data points.

ex.) 29-bar truss (compliance cstr.)

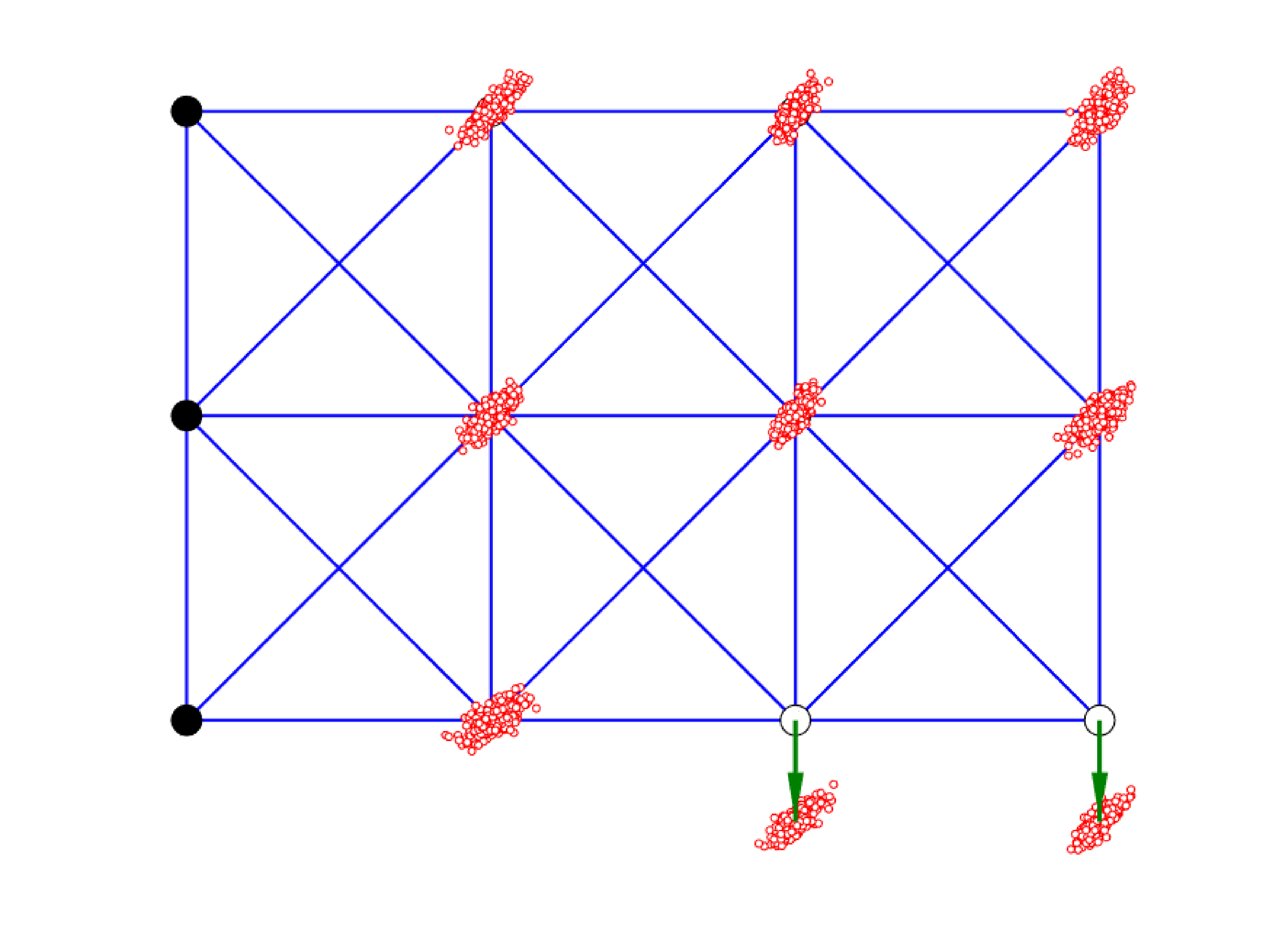
- model :



- compliance constraint

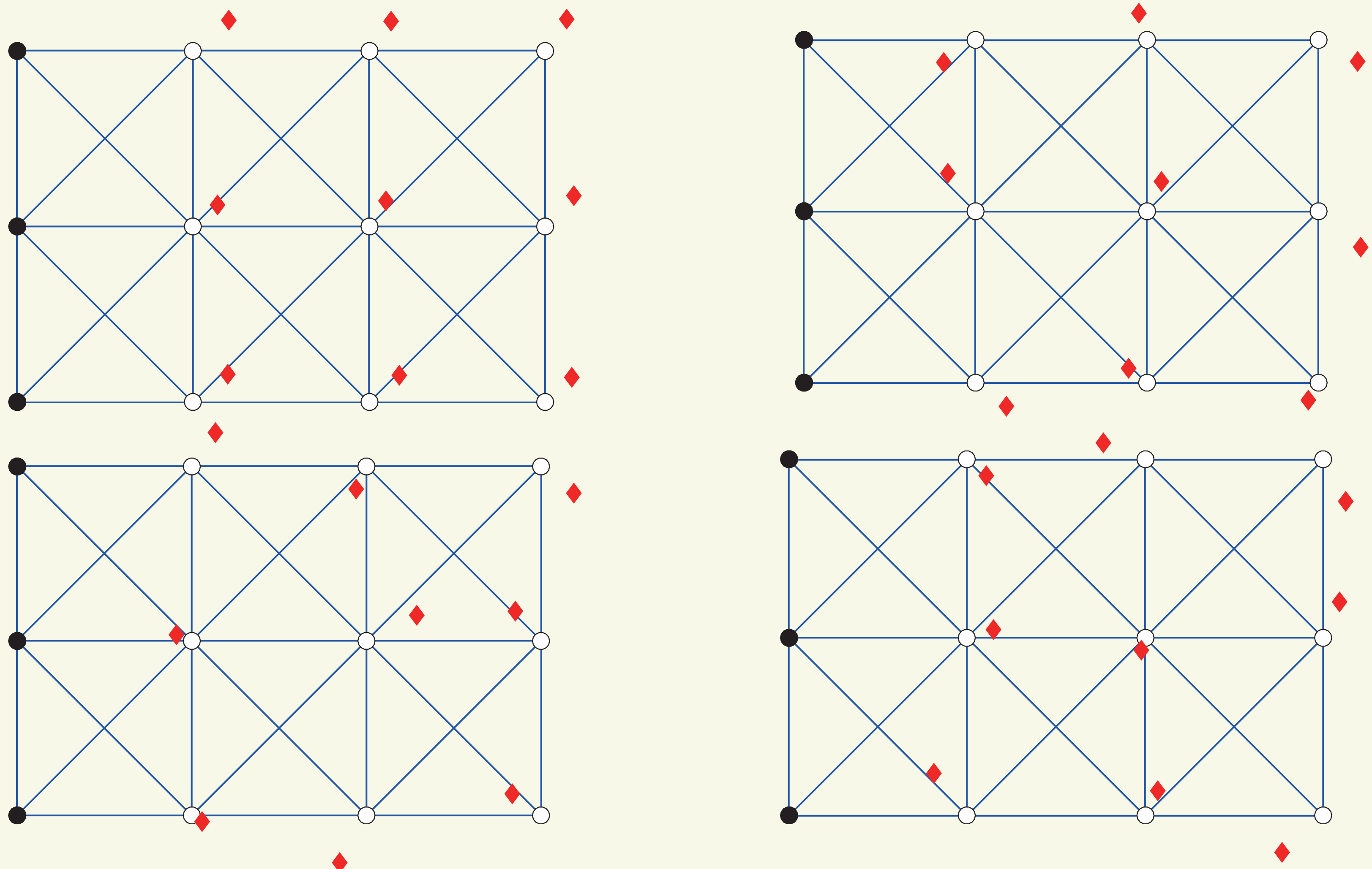
ex.) 29-bar truss (compliance cstr.)

- data of external loads
  - 250 points in $\mathbb{R}^{18}$



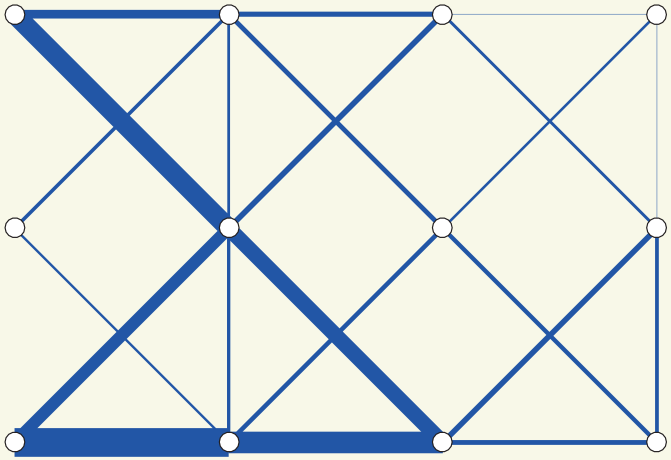
ex.) 29-bar truss (compliance cstr.)

- data of external loads $\xrightarrow[\text{PCA}]{}$ 7 comp. are dominant.
  - first 4 prin. comp.



ex.) 29-bar truss (compliance cstr.)

- 250 data points
  - 50 for shape learning
  - 200 for size calibration
    - 187 points are used for
      $1 - \epsilon = 0.9$ reliability w/ $1 - \delta = 0.92$ confidence



opt. sol. (obj. $= 75.0 \times 10^6\ \mathrm{mm}^3$)

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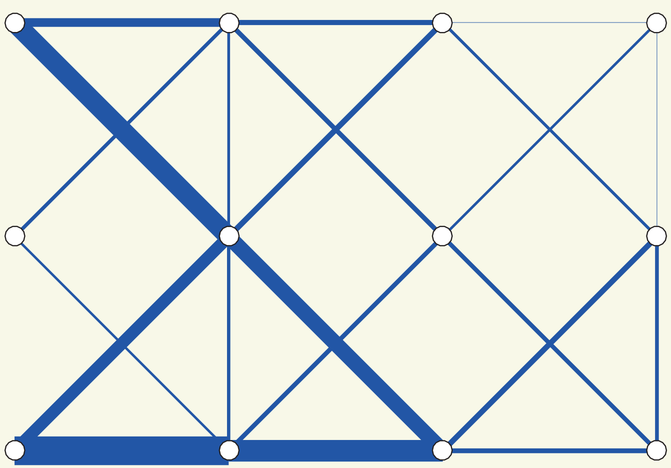

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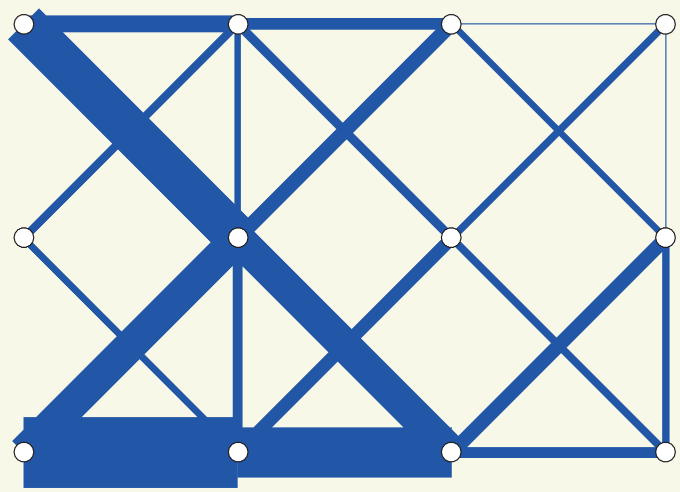
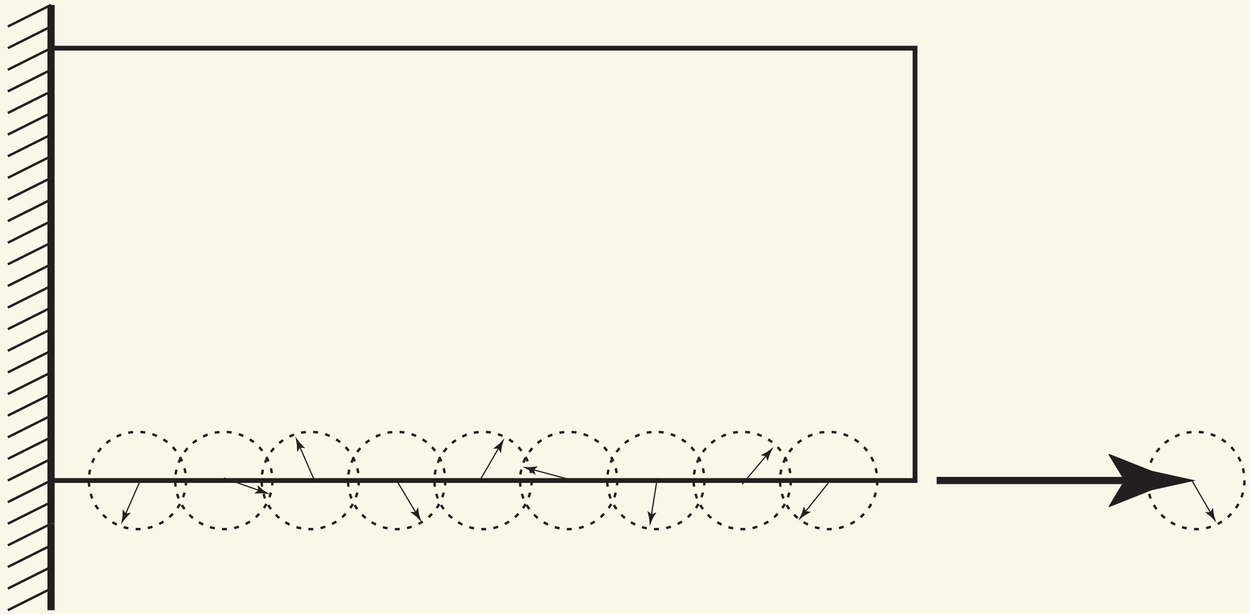


w/o PCA ($176.6 \times 10^6\,\mathrm{mm}^3$)

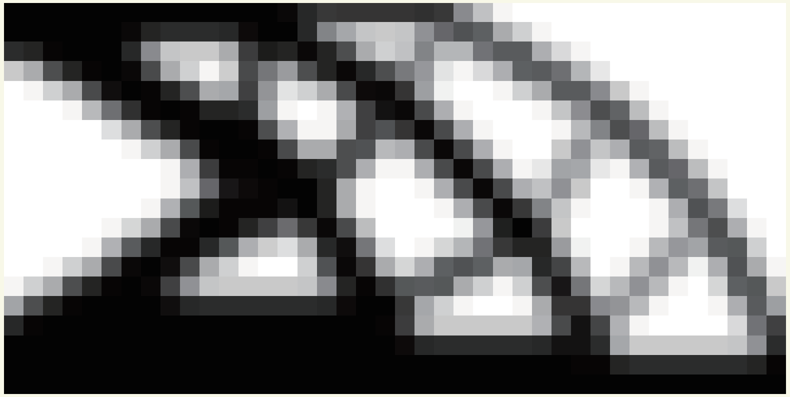
- Over-conservativeness is mitigated by PCA.

ex.) continuum (compliance cstr.)



- 250 data points in $\mathbb{R}^{80}$ $\xrightarrow[\text{PCA}]{}$ 19 dominating components
  - 50 for shape learning
  - 200 for size calibration
    - 187 points are used
      for $1 - \epsilon = 0.9$ reliability w/ $1 - \delta = 0.92$ confidence

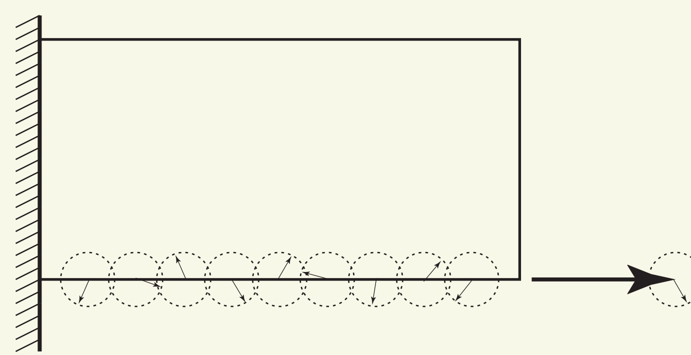
ex.) continuum (compliance cstr.)



opt. sol.
(worst compliance $= 600.0$)

sol. only for mean load
(worst compliance $= 1310.0$)

- volume fraction: specified ($= 0.4852$)
- w/o PCA: infeasible
  ($\because$ “all black sol.” is infeasible.)
  - PCA mitigates over-conservativeness.



conclusions

- uncertainty in input distribution of RBDO
  - reliability w/ confidence level
- reasoning of an assumed uncertainty set in of RO
  - learning from data
- RO conservatively approximating RBDO
  - construction of uncertainty set
    1. dimensionality reduction w/ PCA
    2. shape learning (mean & covariance matrix)
    3. size calibration by order statistics

Many existing methods in RO can be used as tools for RBDO!