Dynamic Steady-State Analysis of Structures under Uncertain Harmonic Loads via Semidefinite Program

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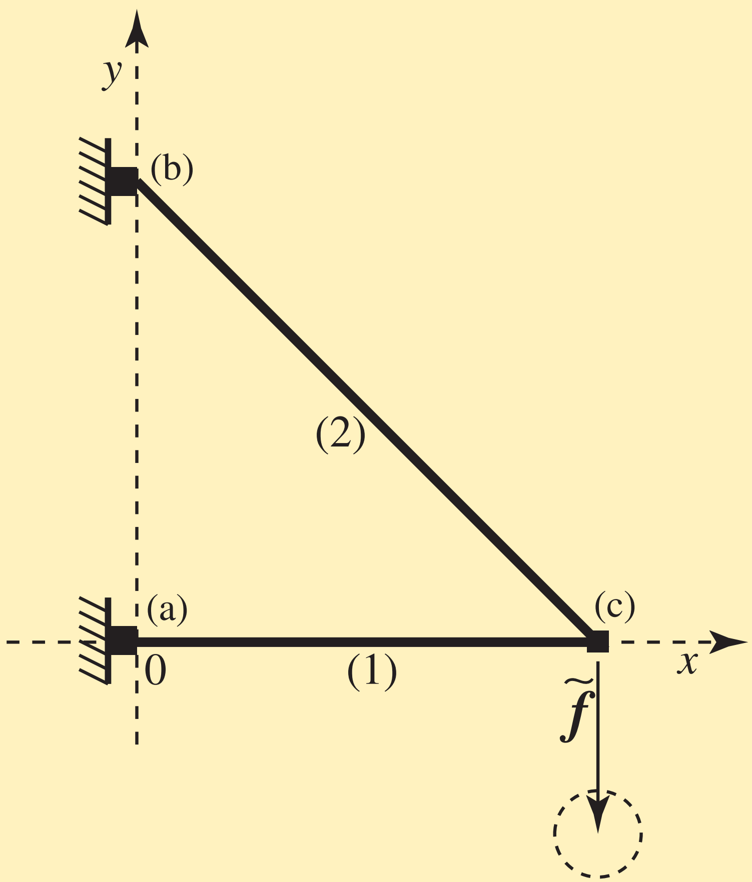
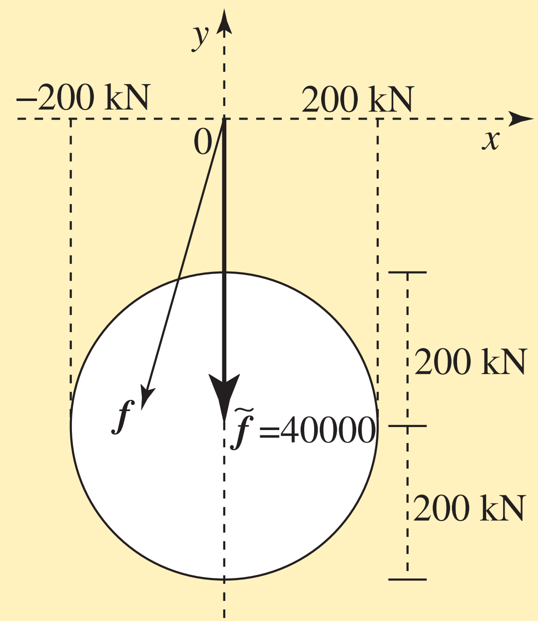
motivation (static analysis)

- 3DOF, 2-bar frame
- uncertain load & stiffness





elastic modulus :

$$180 \leq E_i \leq 220 \text{ GPa}$$

$(24.0 \text{ cm}^2,\ 72.0 \text{ cm}^4)$

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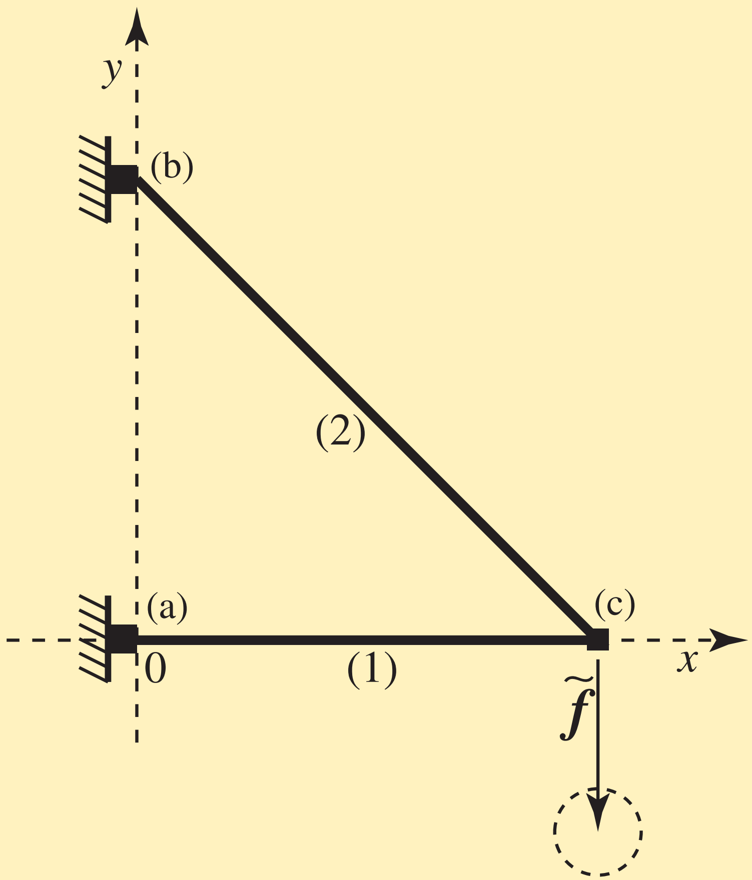
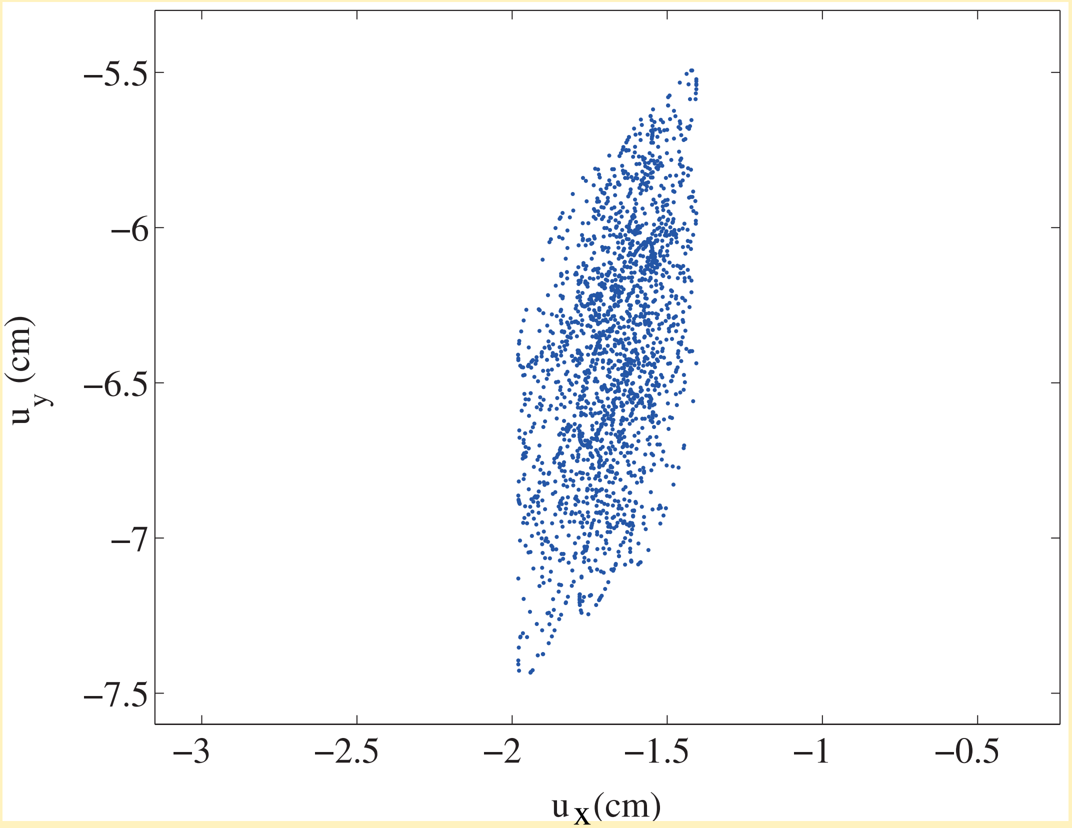
- 3DOF, 2-bar frame [Kanno & Takewaki 06; 08]
- uncertain load & stiffness → bound of response?





distribution of nodal displacement

motivation (static analysis)

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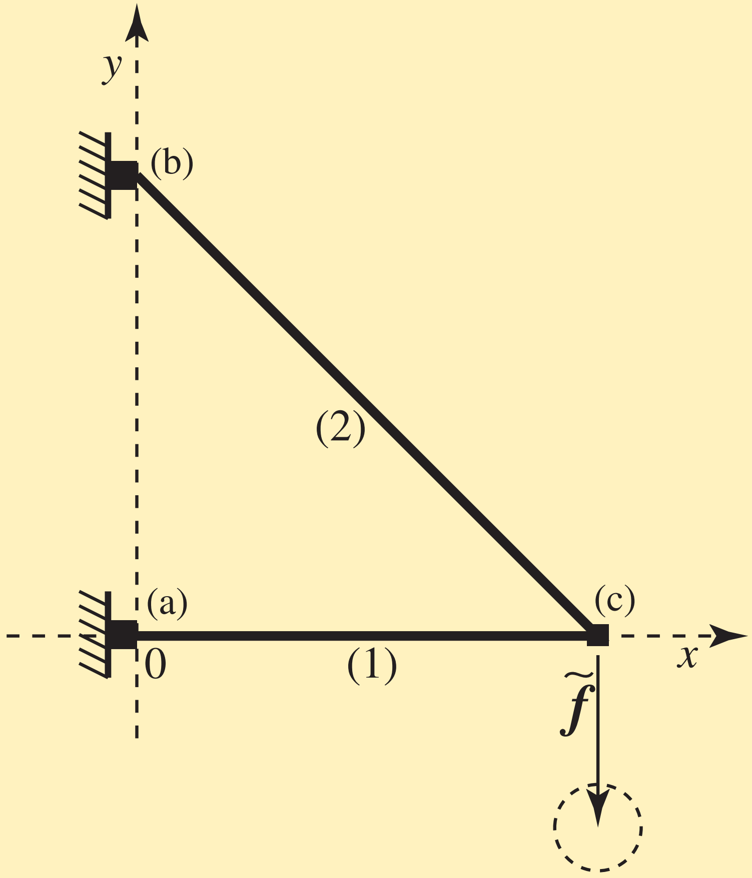

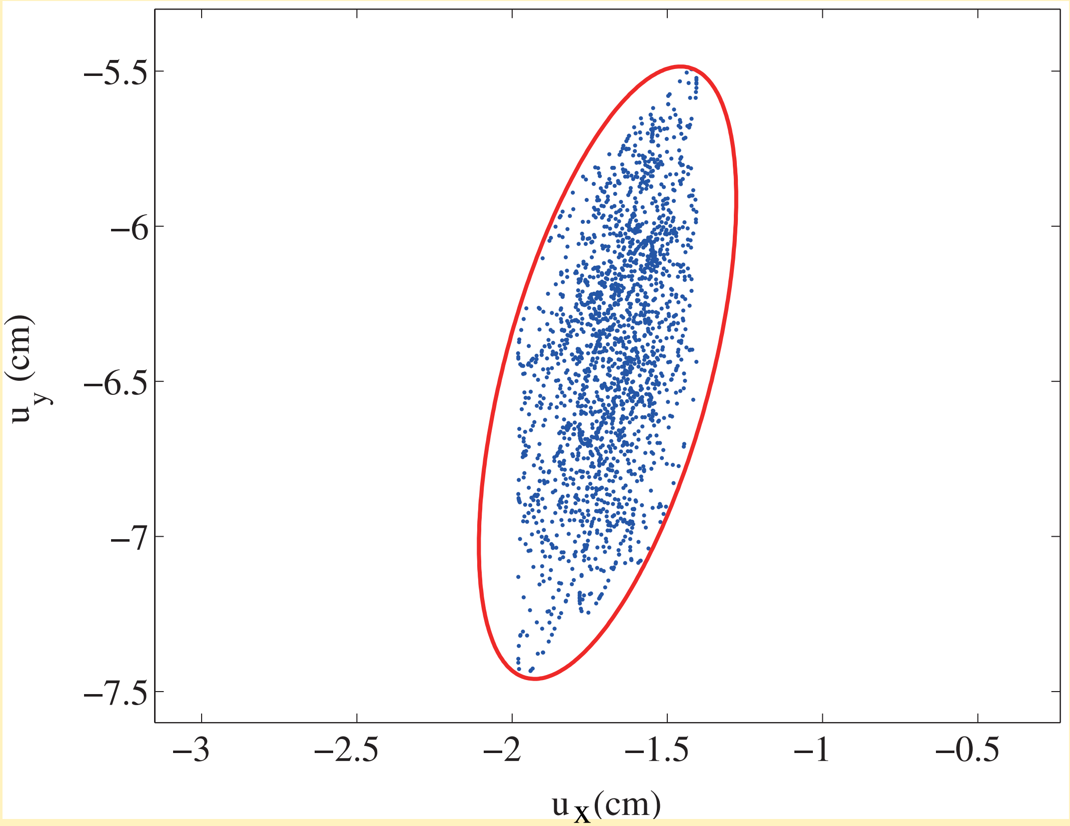

bounding ellipsoid

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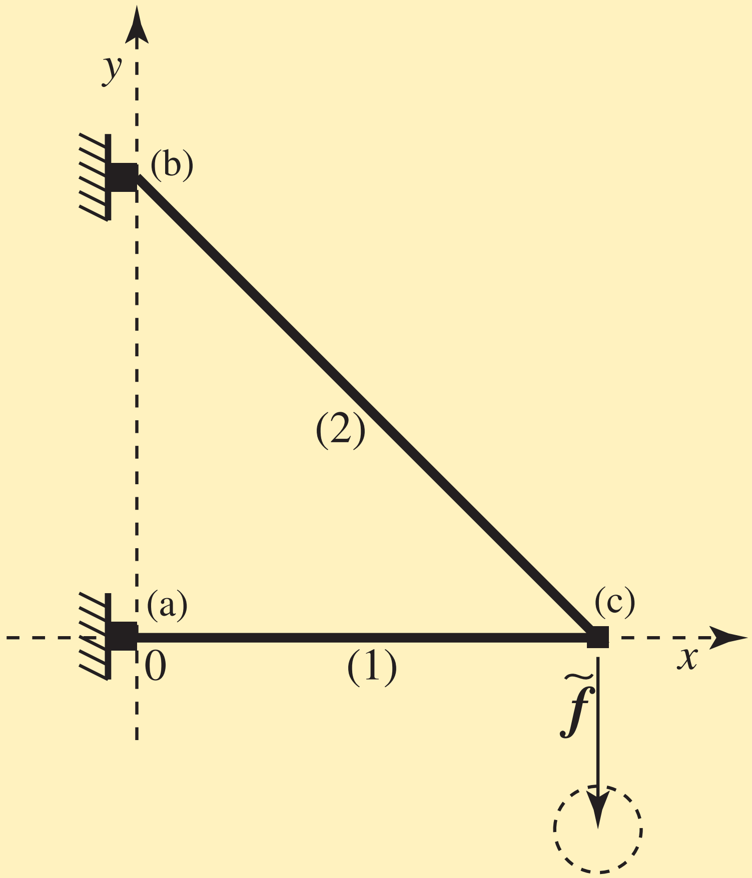

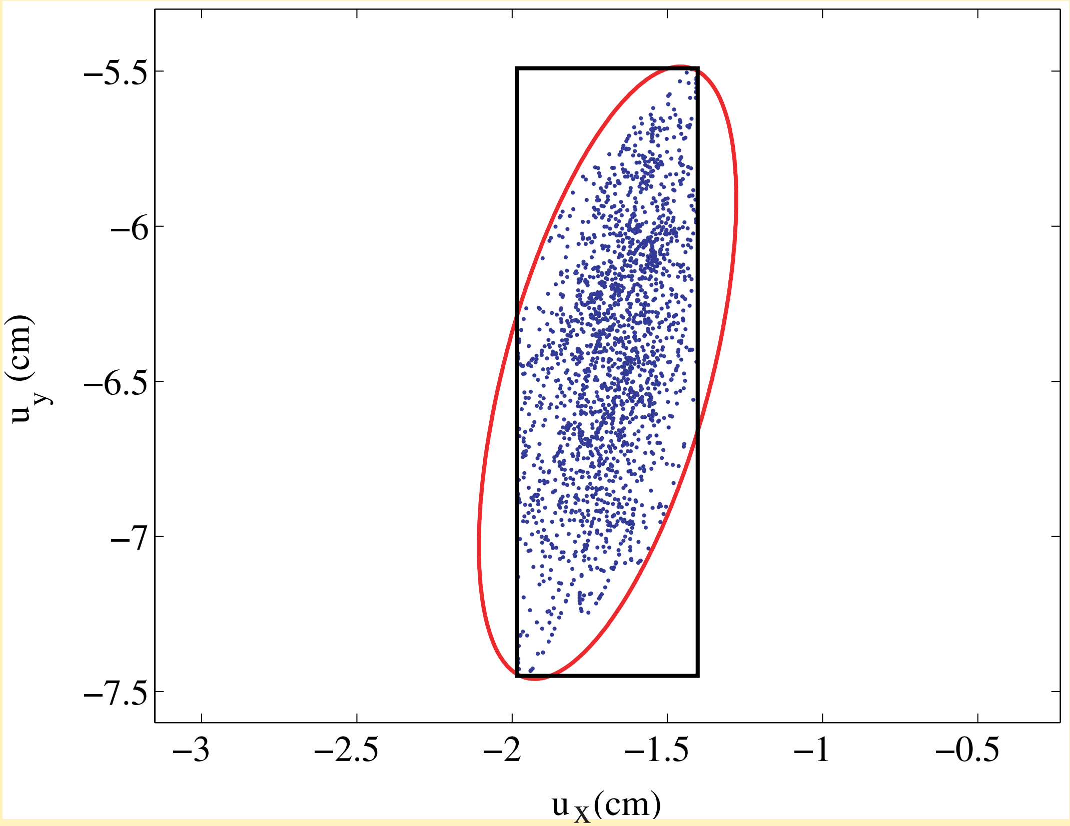

bounding box (interval hull)

uncertainty analysis

- stochastic model — reliability design etc.
- non-stochastic model — unknown-but-bounded parameters
  - convex model [Ben-Haim & Elishakoff 90]
  - interval analysis [Alefeld & Mayer 00], [Chen *et al.* 02], etc.
  - mathematical programming

uncertainty analysis

- stochastic model — reliability design etc.
- non-stochastic model — unknown-but-bounded parameters
  - convex model [Ben-Haim & Elishakoff 90]
    - linear approximation
  - interval analysis [Alefeld & Mayer 00], [Chen *et al.* 02], etc.
    - conservative

      [Neumaier & Pownuk 07] (truss)
  - mathematical programming
    - SDP & proof of conservativeness ← ♣

      [Calafiore & El Ghaoui 04] (ULE)

      [Kanno & Takewaki 06] (static)
    - MIP & extremal case [Guo *et al.* 08], [Kanno & Takewaki 07]

objective: uncertainty analysis

$$\boldsymbol{K}\ddot{\widehat{\boldsymbol{u}}} + \boldsymbol{C}\dot{\widehat{\boldsymbol{u}}} + \boldsymbol{K}\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{f}} \qquad \text{(eq. of motion)}$$

- $\widehat{\boldsymbol{f}} = \mathbf{f} e^{\mathrm{i}\omega t}$ — harmonic excitation
- consider the uncertainty of $\mathbf{f}$
- predict the distribution of $\widehat{\boldsymbol{u}}$
- do not use the 1st order approximation
  $\longrightarrow$ large uncertainties
- have a proof of conservativeness

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mathematical framework

- robust optimization (RO)
- semidefinite program (SDP)
- $\mathcal{S}$-lemma

- confidence bound detection
  $\leftarrow$ optimization / robust optimization

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robust optimization

- nominal (conventional) optimization

$$\min_{\boldsymbol{x}} \{c(\boldsymbol{x}) : \boldsymbol{g}(\boldsymbol{x}) \geq \mathbf{0}\}$$

e.g.) $\boldsymbol{x}$: design variables

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e.g.) $\boldsymbol{x}$: design variables

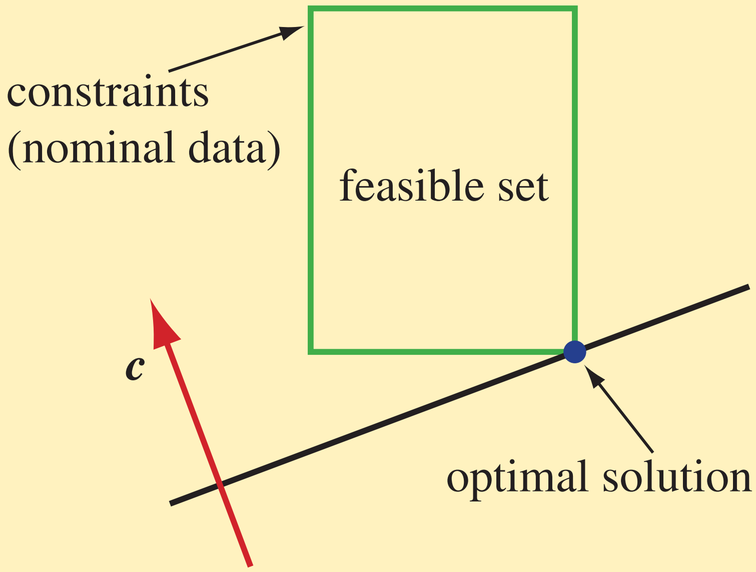
- robust optimization [Ben-Tal & Nemirovski 98; 02]

$$\min_{\boldsymbol{x}}\{c(\boldsymbol{x}) : \boldsymbol{g}(\boldsymbol{x};\boldsymbol{\zeta}) \geq \mathbf{0} \ (\forall \boldsymbol{\zeta} : \alpha \geq \|\boldsymbol{\zeta}\|)\}$$

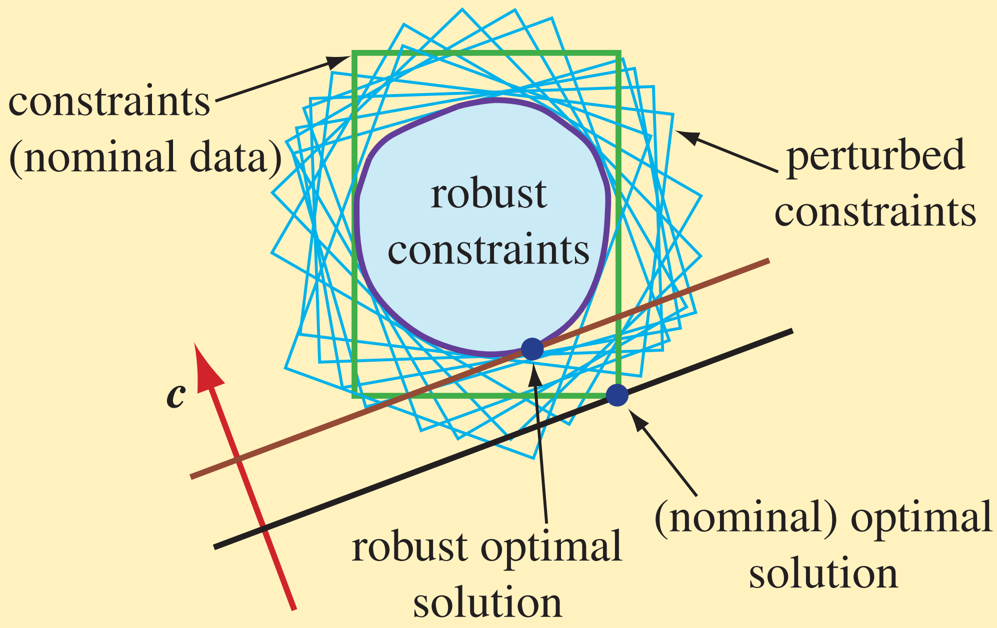
e.g.) $\boldsymbol{\zeta}$: unknown-but-bounded parameters

robust optimization

$$\min_{\boldsymbol{x}}\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x};\boldsymbol{\zeta}) \geq \mathbf{0}\ (\forall\boldsymbol{\zeta} : \alpha \geq \|\boldsymbol{\zeta}\|)\}$$



nominal optimization



robust optimization

semidefinite program (SDP)

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \boldsymbol{C} - \sum_{i=1}^{m} \boldsymbol{A}_i y_i \succeq \boldsymbol{O} \end{aligned}$$

variables : $y_1, \dots, y_m$

coefficients : $b_1, \dots, b_m,$

$\boldsymbol{A}_1, \dots, \boldsymbol{A}_m,\ \boldsymbol{C} \in \mathcal{S}^n$ $n \times n$ symmetric matrices

- $\boldsymbol{P} \succeq \boldsymbol{O}$ ($\Leftrightarrow$ $\boldsymbol{P}$ is positive semidefinite) $\leftarrow$ nonlinear, convex
- primal-dual interior-point method
- applications in eigenvalue optimization, etc.

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uncertainty model

$$\left(-\omega^2 \boldsymbol{M} + \mathrm{i}\omega \boldsymbol{C} + \boldsymbol{K}\right) \boldsymbol{u} = \mathbf{f} \qquad \text{(eq. of motion)}$$

- $\widehat{\boldsymbol{f}} = \mathbf{f} e^{\mathrm{i}\omega t}$ — harmonic excitation

uncertainty:

$$\mathbf{f} = \widetilde{\boldsymbol{f}}_j + \boldsymbol{F}_0 \zeta, \quad \zeta \in \mathcal{Z}$$

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$\widetilde{\boldsymbol{f}}$	nominal (best estimate) value
ζ	unknown-but-bounded
$\boldsymbol{F}_0$	coefficients ('magnitudes' of uncertainties)
$\mathcal{Z}$	closed set

uncertainty model

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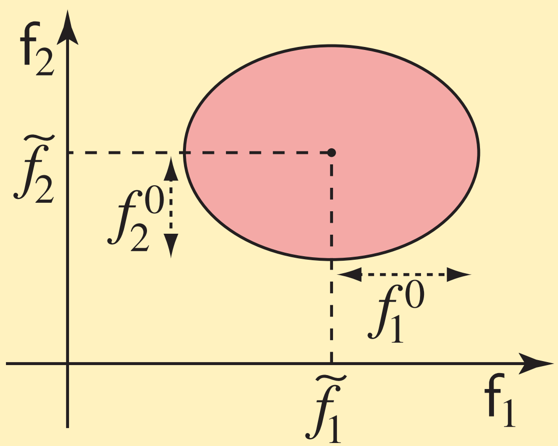
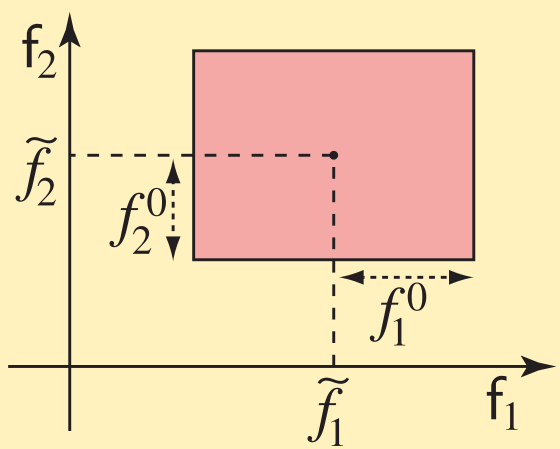


$\mathcal{Z} = \{\zeta \mid 1 \geq \|\boldsymbol{T}_j \zeta\|_2 \; (\forall j)\}$



$\mathcal{Z} = \{\zeta \mid 1 \geq \|\zeta\|_\infty\}$

- ‘magnitude & direction’ of amplitude of driving load are uncertain

reduction to real variables

$$\left(-\omega^2 \boldsymbol{M} + \mathrm{i}\omega \boldsymbol{C} + \boldsymbol{K}\right) \boldsymbol{u} = \mathbf{f} \qquad \text{(eq. of motion)}$$

- $\boldsymbol{u} \in \mathbb{C}^d$ (complex vector)

$$\begin{bmatrix} -\omega^2 \boldsymbol{M} + \boldsymbol{K} & -\omega \boldsymbol{C} \\ \omega \boldsymbol{C} & -\omega^2 \boldsymbol{M} + \boldsymbol{K} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

reduction to real variables

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- $\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\, \boldsymbol{u} \\ \mathrm{Im}\, \boldsymbol{u} \end{bmatrix}$ (: real part) (: imag. part) is a real vector

- for u_q (displacement amplitude):
  - $|u_q|$ (modulus) — $r = \sqrt{v_{1q}^2 + v_{2q}^2}$
  - $\arg u_q$ (argument) — $\tan\theta = v_{2q}/v_{1q}$

upper bound of modulus

maximum value of modulus (def.)

$$r_{\max} = \max\{\|\boldsymbol{v}_q\| \mid \boldsymbol{v} \text{ solves } (\spadesuit)\} \qquad \boldsymbol{v}_q = (v_{1q}, v_{2q})^{\mathrm{T}}$$

eq. of motion (w.r.t. real variables)

$$\boldsymbol{S}\boldsymbol{v} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{f} \in \mathcal{F} \qquad (\spadesuit)$$

- $\max\{\|\boldsymbol{v}_q\|\}$ — nonconvex optimization
- direct use of nonlinear programming
  $\longrightarrow$ no proof of conservativeness

upper bound of modulus

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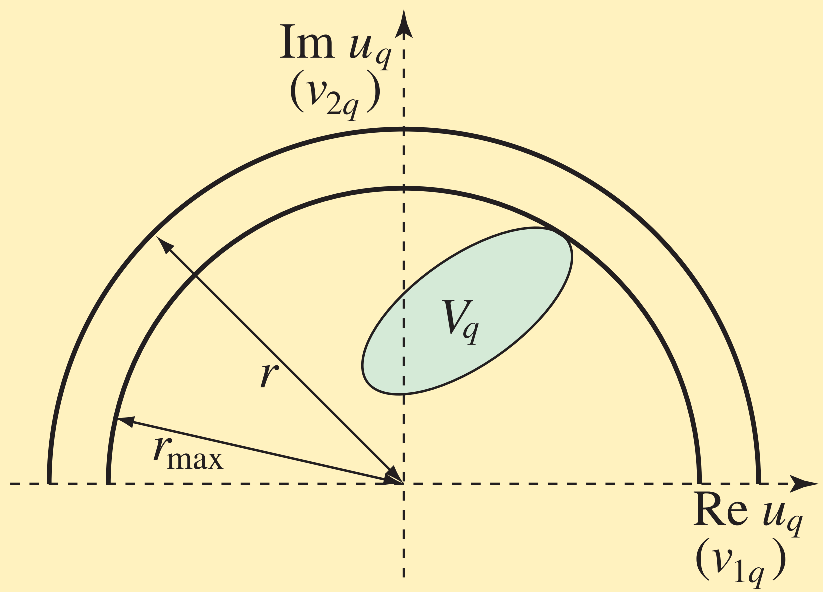
reformulation to robust optimization

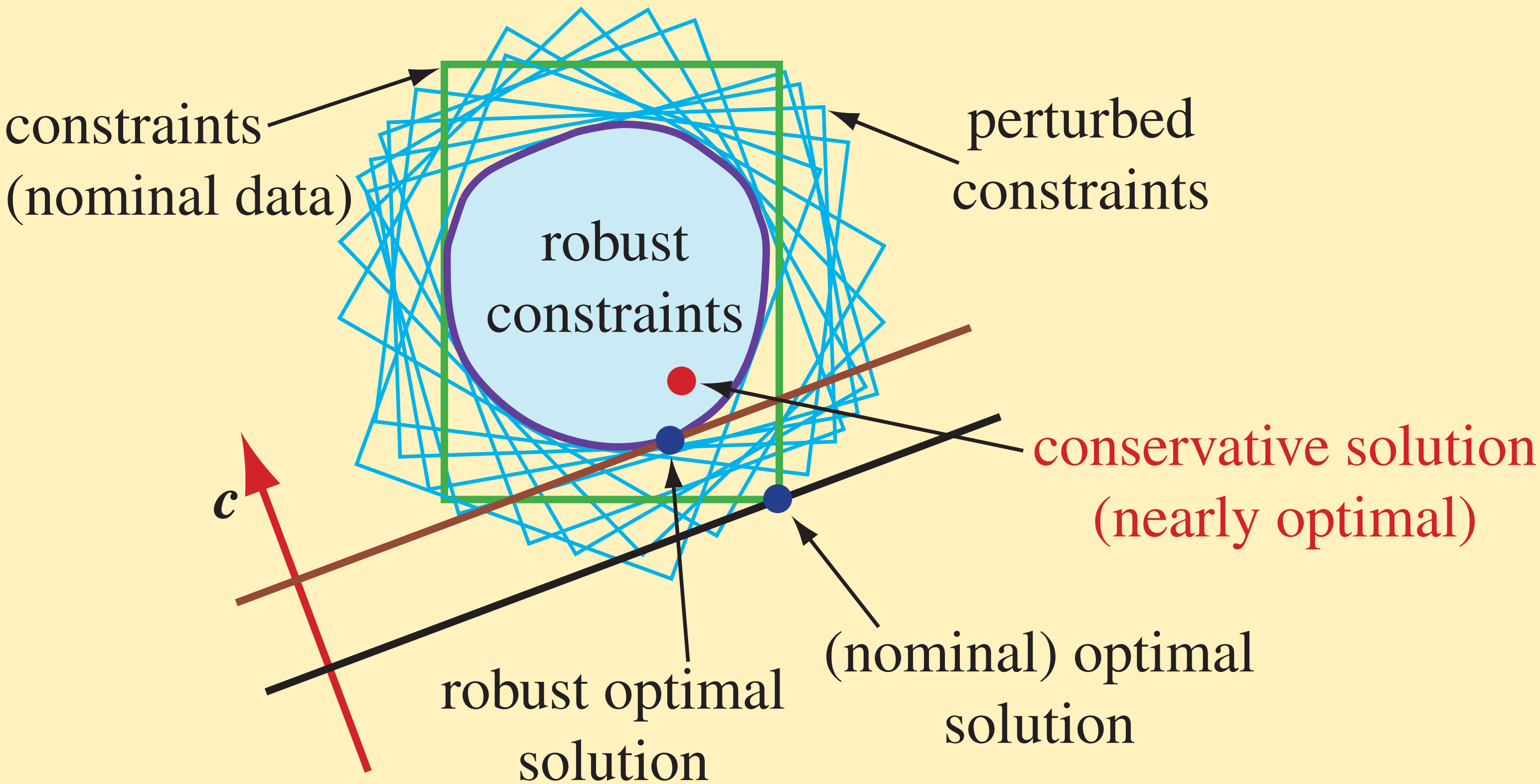
$$r_{\max} = \min\left\{r \mid r \geq \|\boldsymbol{v}_q\| \ (\forall \boldsymbol{v} \in \mathcal{V})\right\} \tag{RO}$$



conservative method for (RO) $\longrightarrow$ find an upper bound of $r_{\max}$

cf) robust optimization

$$\min_{\boldsymbol{x}} \{ \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{\zeta}) \geq \mathbf{0} \ (\forall \boldsymbol{\zeta} : \alpha \geq \|\boldsymbol{\zeta}\|) \}$$



- SDP relaxation technique
  $\longrightarrow$ find a conservative solution

$\mathcal{S}$-lemma

$f(\boldsymbol{x}), g(\boldsymbol{x})$: quadratic functions

(a): $f(\boldsymbol{x}) \geq 0 \quad \Longrightarrow \quad g(\boldsymbol{x}) \geq 0$

$\Updownarrow$

(b): $\exists w \geq 0, \quad g(\boldsymbol{x}) \geq w f(\boldsymbol{x}), \ \forall \boldsymbol{x}$

$\mathcal{S}$-lemma

$f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x}), g(\boldsymbol{x})$: quadratic functions

(a): $f_1(\boldsymbol{x}) \geq 0, \dots, f_m(\boldsymbol{x}) \geq 0 \quad \Longrightarrow \quad g(\boldsymbol{x}) \geq 0$

$\Uparrow$

(b): $\exists \boldsymbol{w} \geq \boldsymbol{0}, \quad g(\boldsymbol{x}) \geq \sum_{i=1}^{m} w_i f_i(\boldsymbol{x}), \ \forall \boldsymbol{x}$

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- cf.) Farkas' lemma ($f_1, \dots, f_m, g$: linear functions)

(a) $\Leftrightarrow$ the system $f_1(\boldsymbol{x}) \geq 0, \dots, f_m(\boldsymbol{x}) \geq 0, \ g(\boldsymbol{x}) < 0$ does not have a solution

$\mathcal{S}$-lemma

$f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x}), g(\boldsymbol{x})$: quadratic functions

(a): $f_1(\boldsymbol{x}) \geq 0, \dots, f_m(\boldsymbol{x}) \geq 0 \quad \Longrightarrow \quad g(\boldsymbol{x}) \geq 0$

$\Uparrow$

(b): $\exists \boldsymbol{w} \geq \boldsymbol{0}, \quad g(\boldsymbol{x}) \geq \sum_{i=1}^{m} w_i f_i(\boldsymbol{x}), \ \forall \boldsymbol{x}$ $\rightarrow$ p.s.d. constraint of SDP

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(b): $\exists \boldsymbol{w} \geq \boldsymbol{0}, \quad g(\boldsymbol{x}) \geq \sum_{i=1}^{m} w_i f_i(\boldsymbol{x}), \ \forall \boldsymbol{x}$

- apply $\mathcal{S}$-lemma to

$$r_{\max} = \min \{ r \mid r \geq \|\boldsymbol{v}_q\| \ (\forall \boldsymbol{v} \in \mathcal{V}) \} \qquad \text{(RO)}$$

- obtain an SDP providing $r^* \ (\geq r^{\max})$ (conservative bound)

bounds of argument

maximum value of argument (def.)

$$\theta_{\max} = \max\left\{\arctan(v_{1q}/v_{2q}) \mid \boldsymbol{v} \in \mathcal{V}\right\}$$

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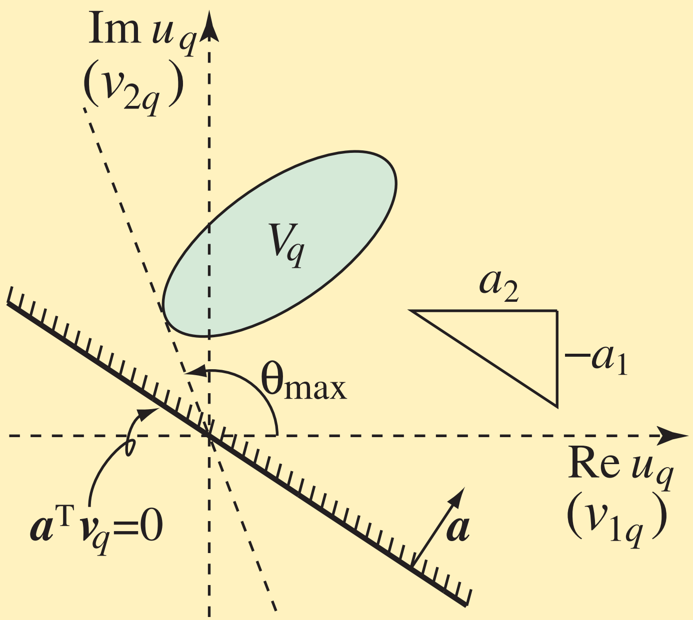

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} v_{1q} \\ v_{2q} \end{bmatrix} = 0 \quad \Rightarrow \quad \frac{v_{1q}}{v_{2q}} = -\frac{a_1}{a_2}$$

bounds of argument

maximum value of argument (def.)

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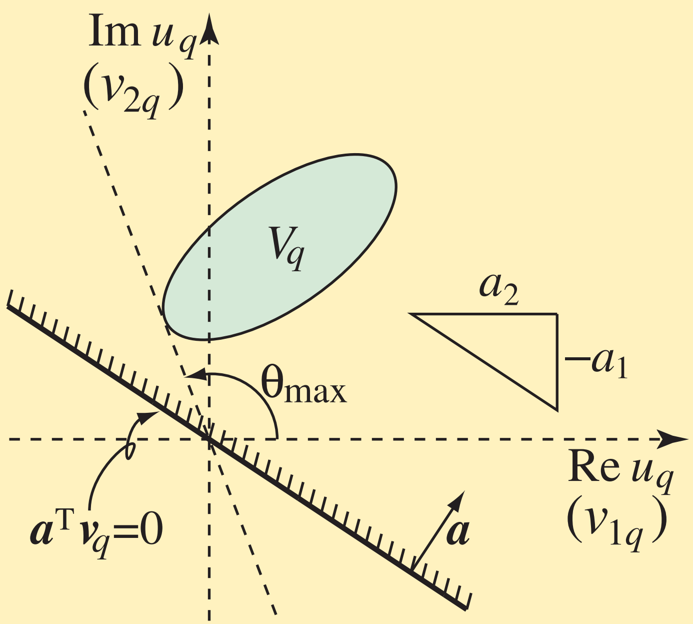
reformulation to robust optimization

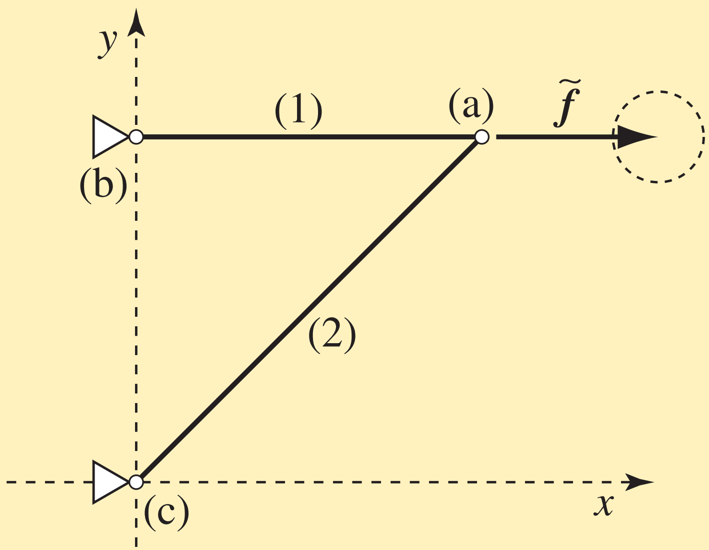
$$-1/\tan\theta_{\max} = \min\left\{a_2 \mid \begin{bmatrix}1 & a_2\end{bmatrix}\begin{bmatrix}v_{1q} \\ v_{2q}\end{bmatrix} \geq 0 \ (\forall \boldsymbol{v} \in \mathcal{V})\right\} \qquad \text{(RO)}$$

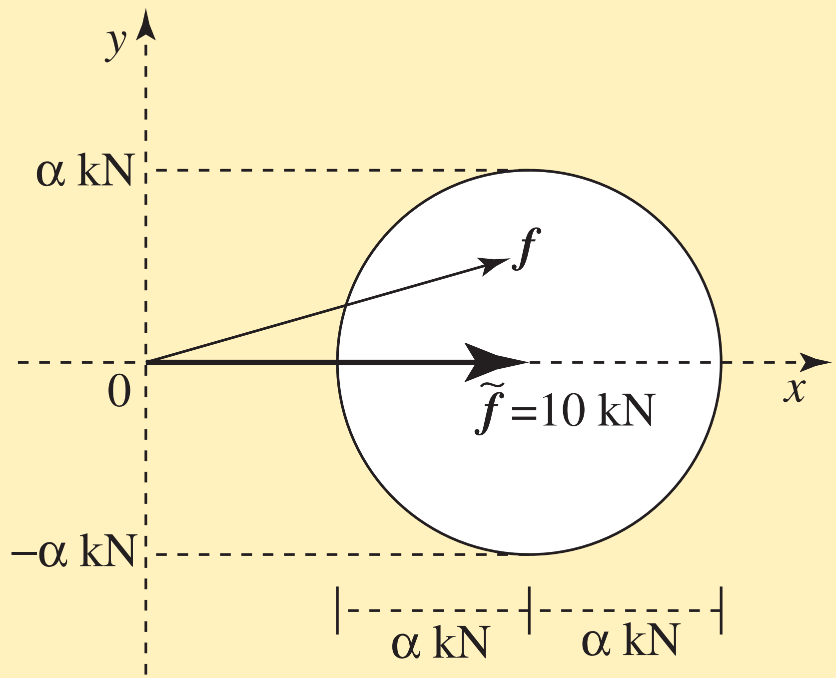


conservative method for (RO) $\longrightarrow$ find an upper bound of $\theta_{\max}$

ex. (2DOF, 2-bar truss)

- harmonic driving load — $\boldsymbol{f} e^{\mathrm{i}\omega t}$
- uncertainty — $\boldsymbol{f} = \tilde{\boldsymbol{f}} + \boldsymbol{\zeta} \quad \alpha \geq \|\boldsymbol{\zeta}\|$
- $\alpha = 10\%$





- primal-dual interior-point method: SeDuMi 1.05 [Sturm 99]

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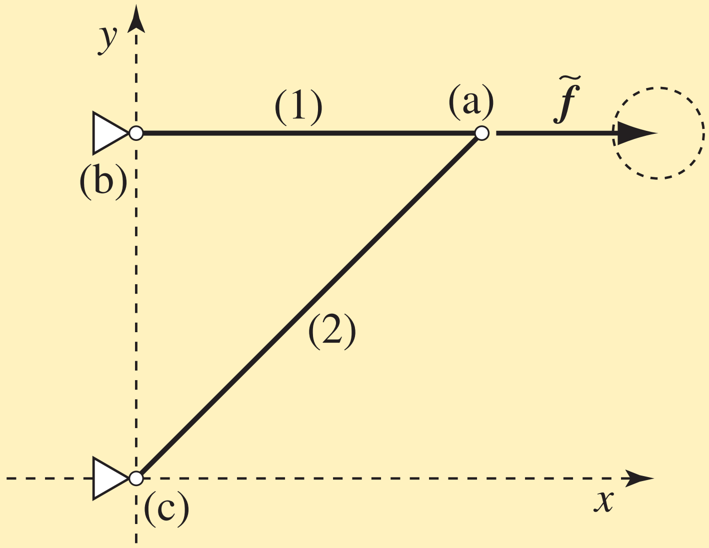
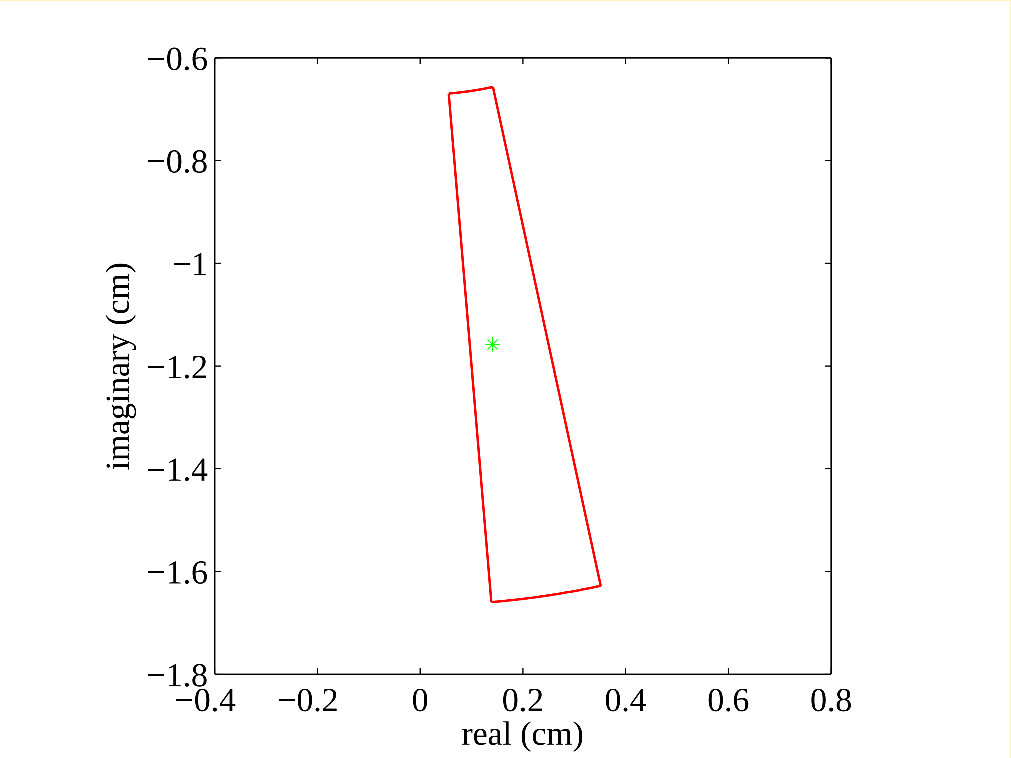
- uncertainty — $\boldsymbol{f} = \widetilde{\boldsymbol{f}} + \boldsymbol{\zeta} \quad \alpha \geq \|\boldsymbol{\zeta}\|$
- $\alpha = 10\% \qquad \omega = \omega_1^0$: fundamental circular frequency (undamped)





displacement amplitude $u_x (\in \mathbb{C})$

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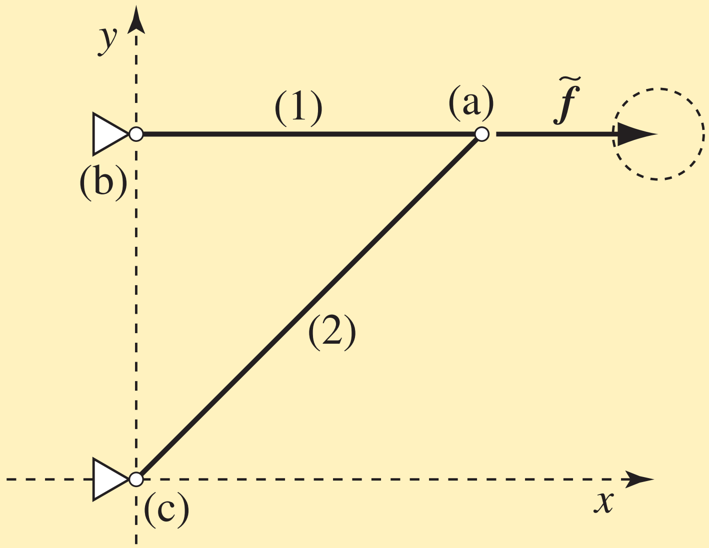

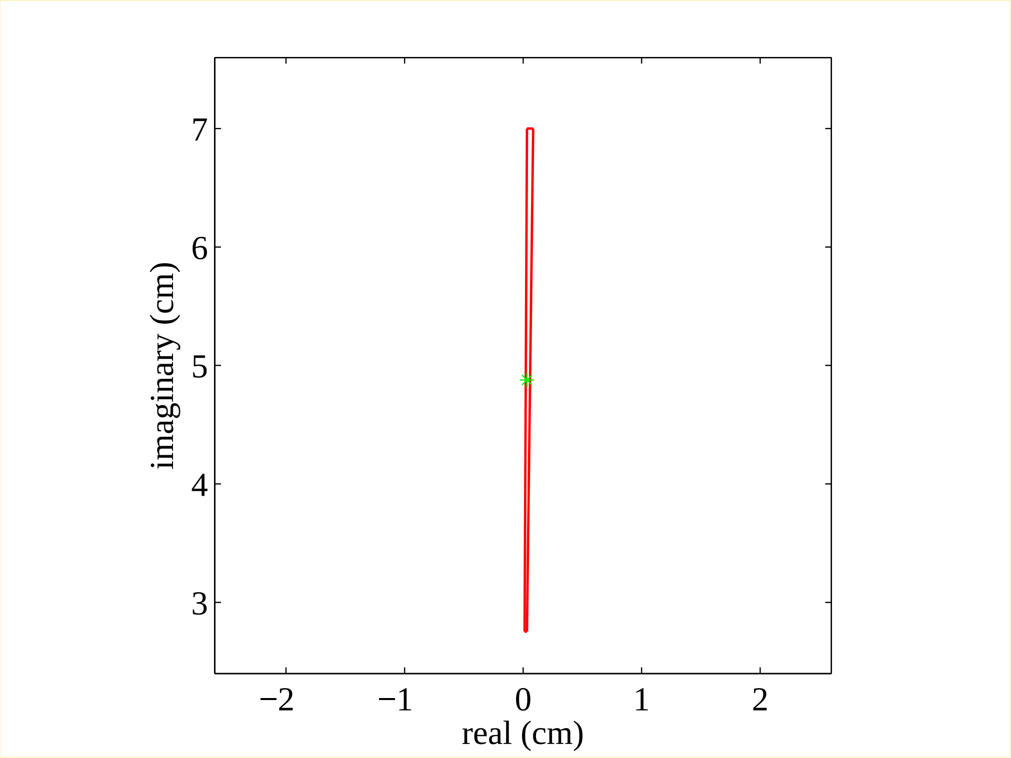

displacement amplitude $u_y (\in \mathbb{C})$

ex. (2DOF, 2-bar truss)

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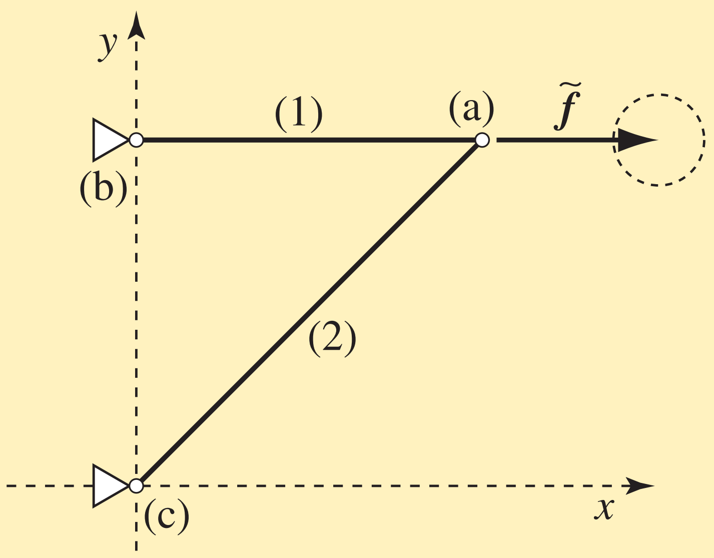
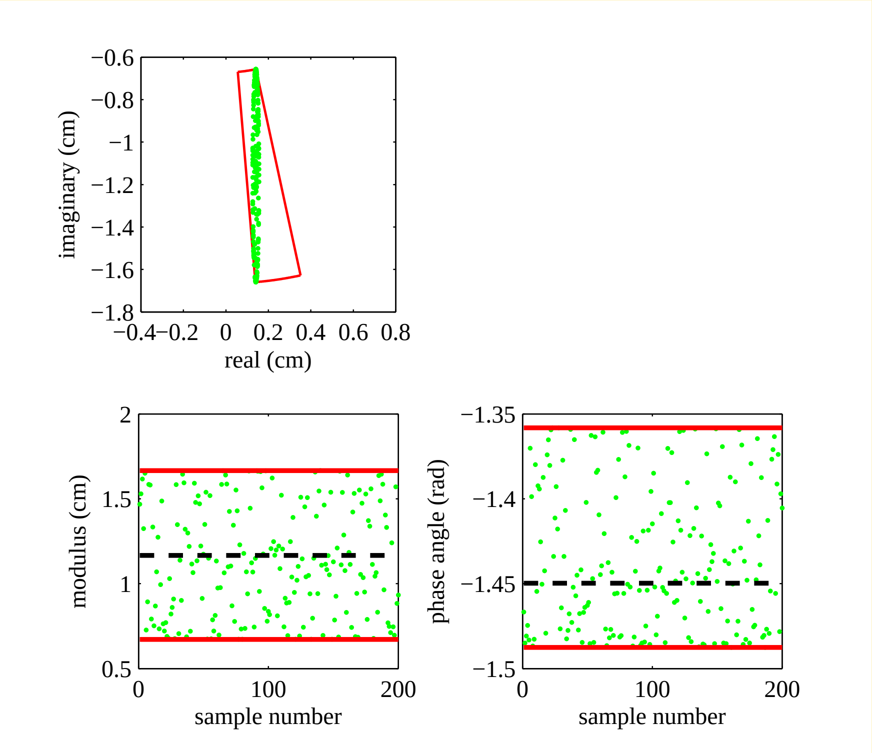
- uncertainty — $\boldsymbol{f} = \widetilde{\boldsymbol{f}} + \boldsymbol{\zeta} \quad \alpha \geq \|\boldsymbol{\zeta}\|$
- $\alpha = 10\% \qquad \omega = \omega_1^0$: fundamental circular frequency (undamped)

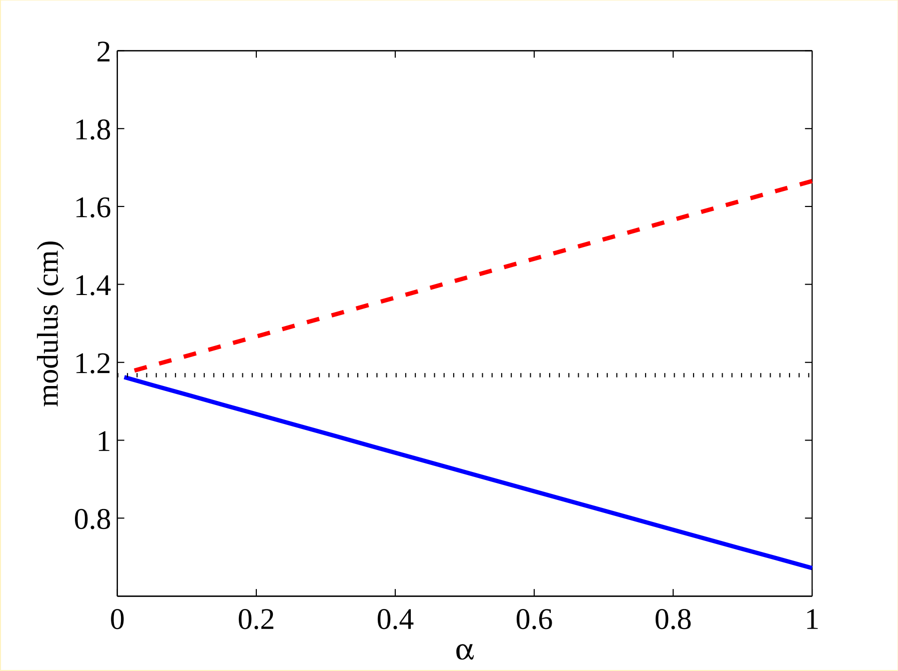




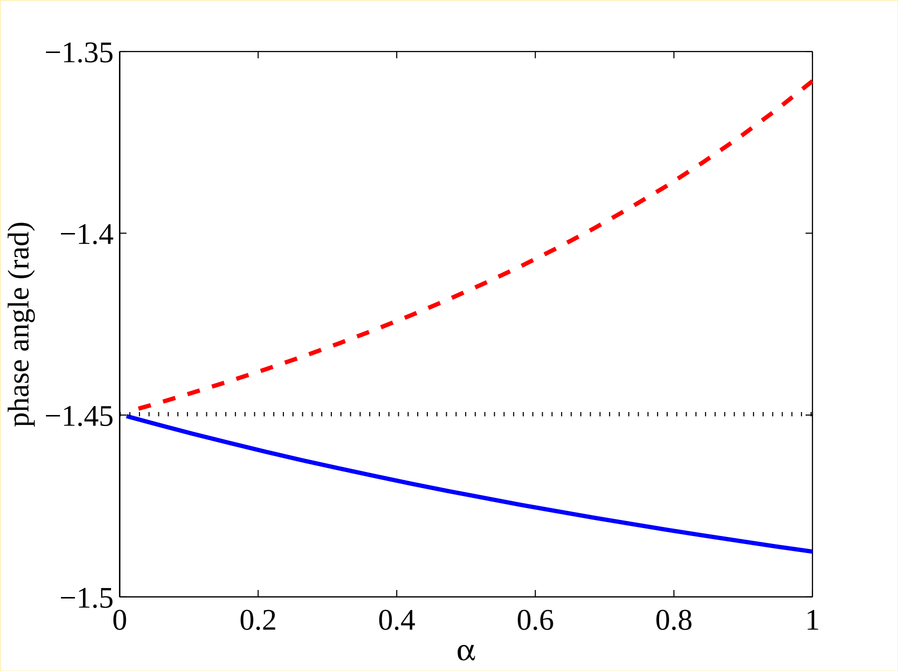
random samples (u_x)

ex. (2DOF, 2-bar truss)

- $(\omega = \omega_1^0)$ variations w.r.t. α ('magnitude' of uncertainty)



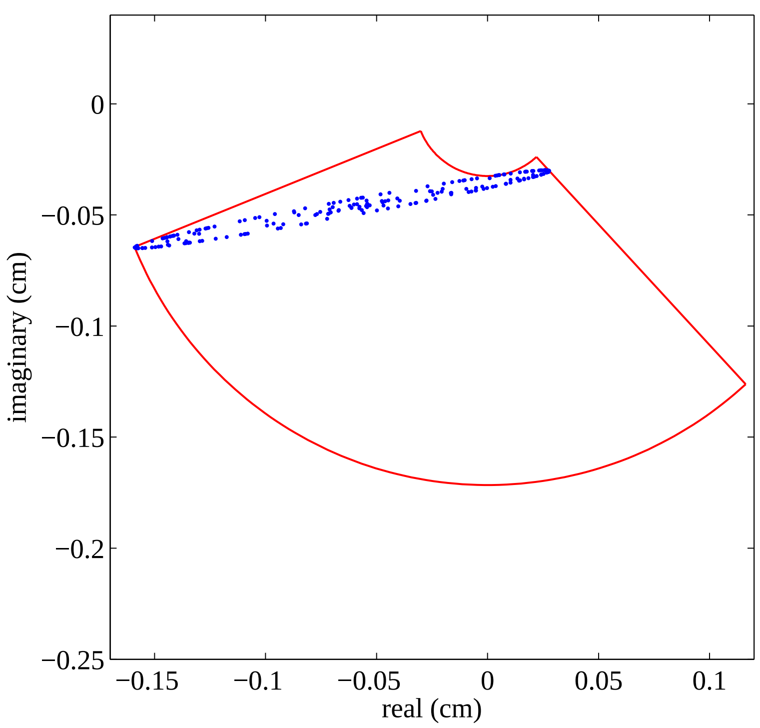
$|u_x|$



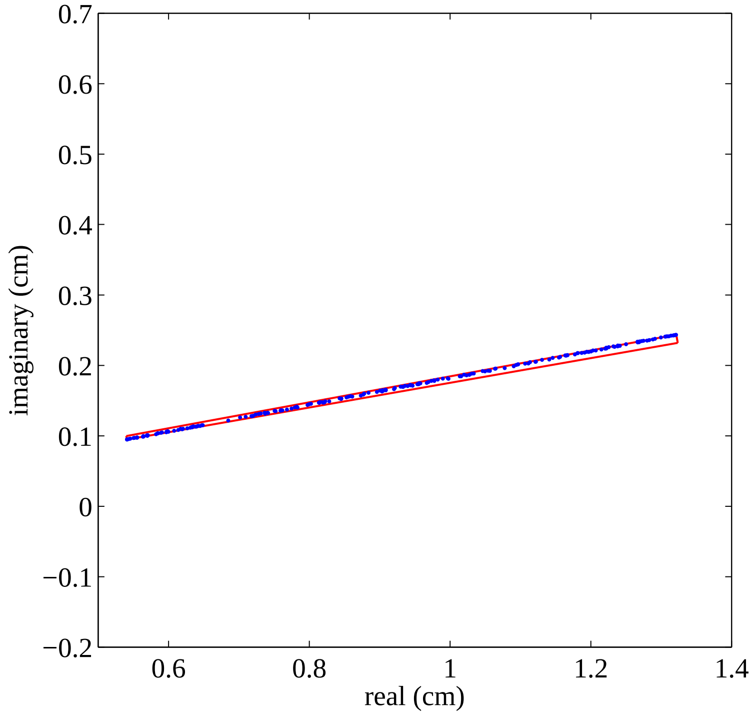
$\arg u_x$

ex. (2DOF, 2-bar truss)

- driving load $\boldsymbol{f}e^{\mathrm{i}\omega t}$ $(\omega = 1.1\omega_1^0)$



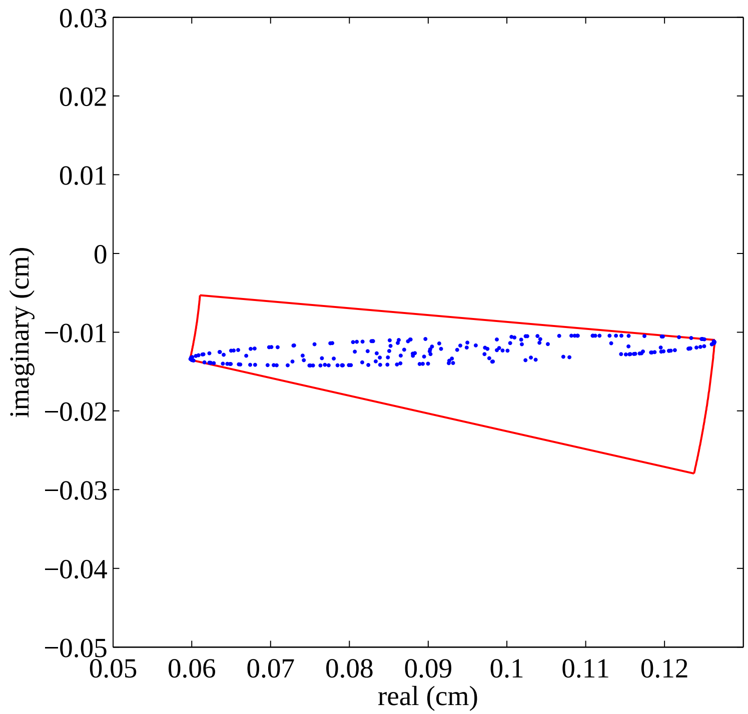
displacement amplitude u_x

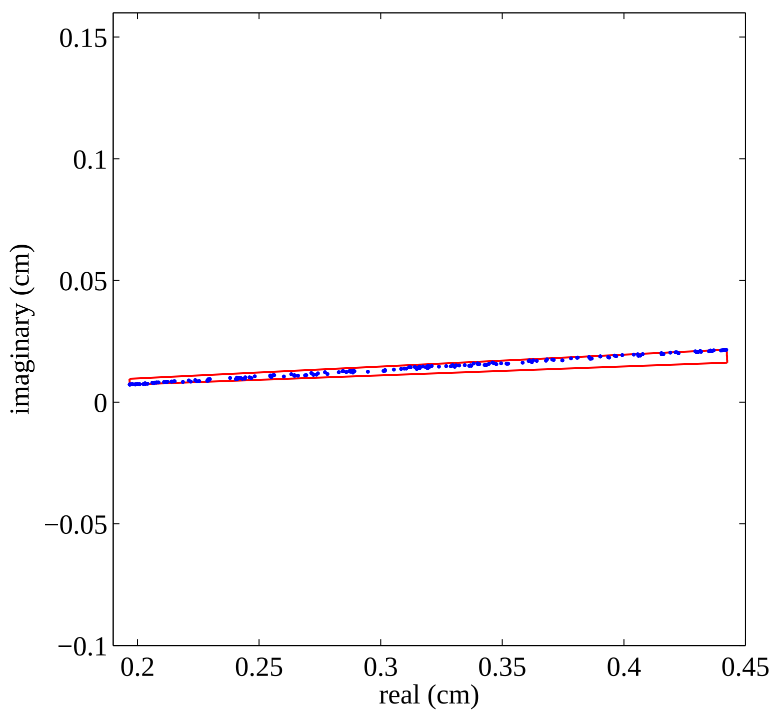


displacement amplitude u_y

ex. (2DOF, 2-bar truss)

- $(\omega = 1.3\omega_1^0)$

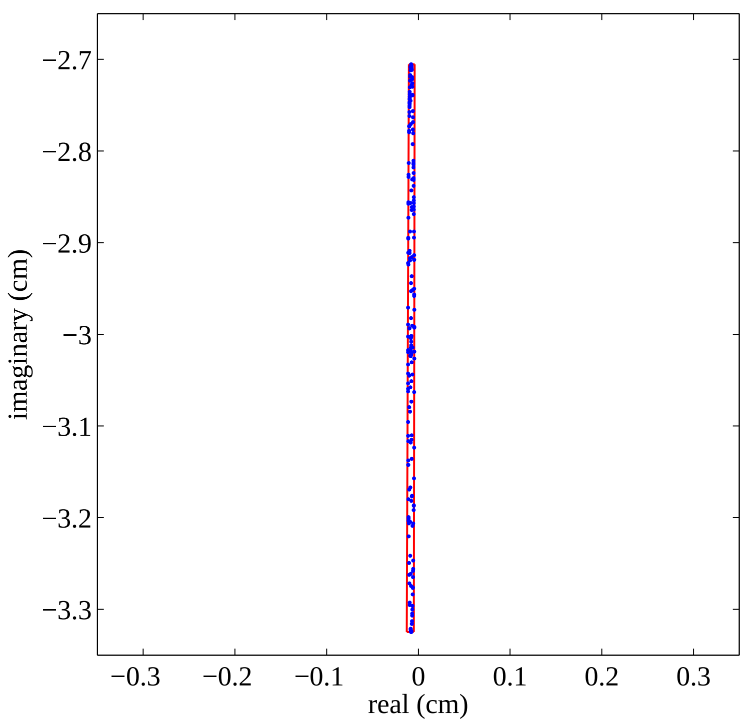




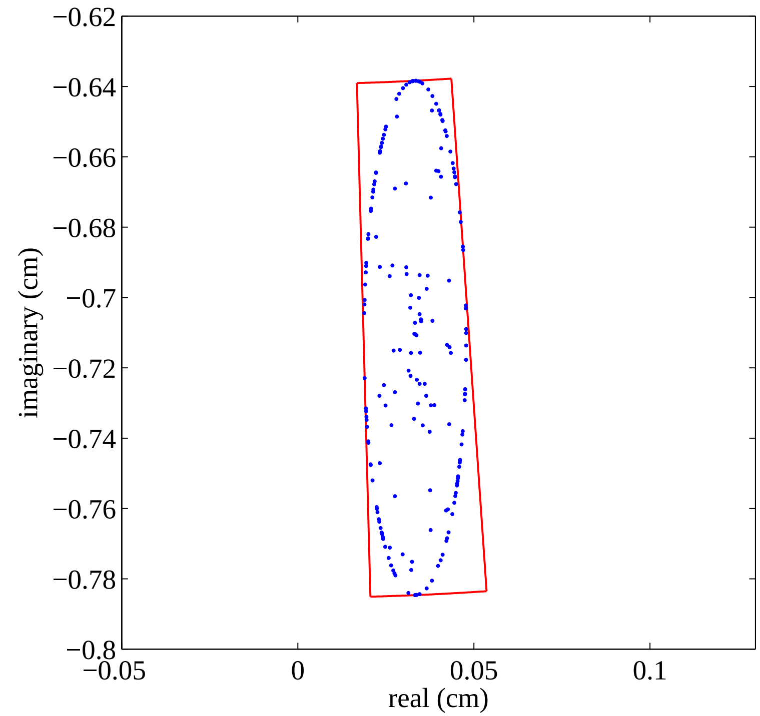
displacement amplitude u_x displacement amplitude u_y

ex. (2DOF, 2-bar truss)

- $(\omega = \omega_2^0)$: 2nd fundamental frequency (undamped)



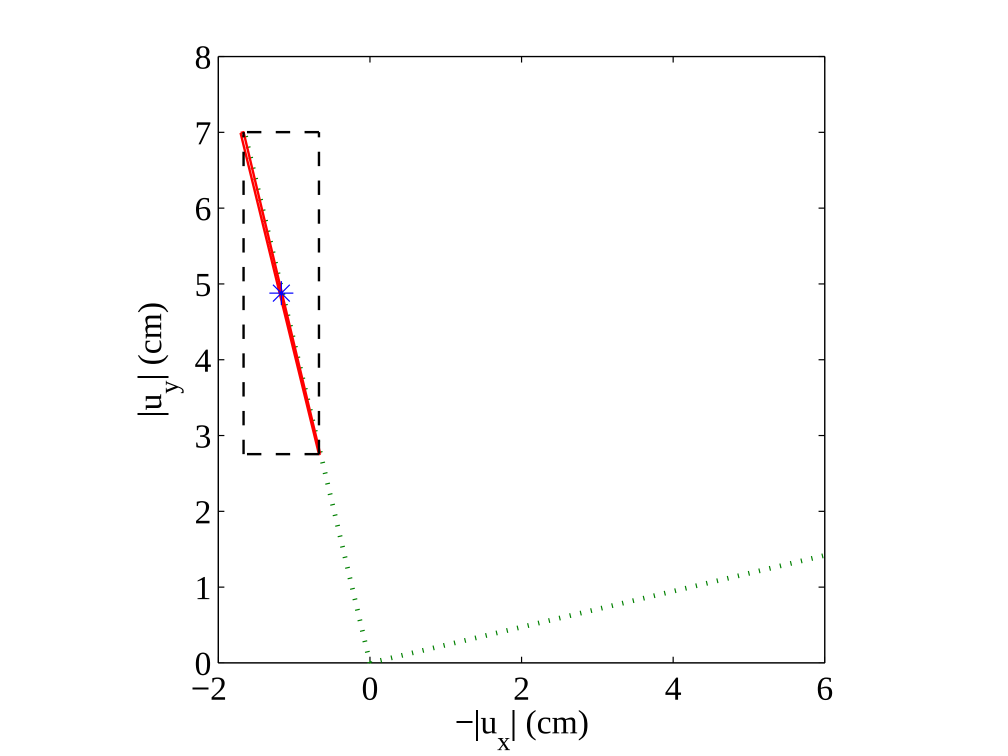
displacement amplitude u_x

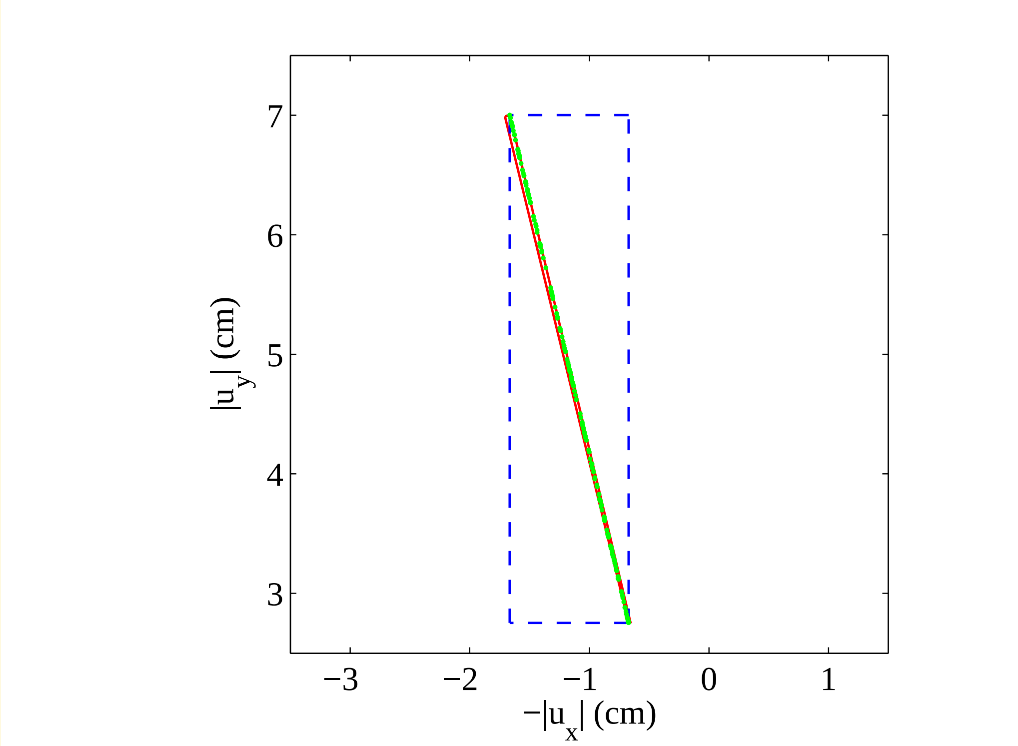


displacement amplitude u_y

ex. (2DOF, 2-bar truss)

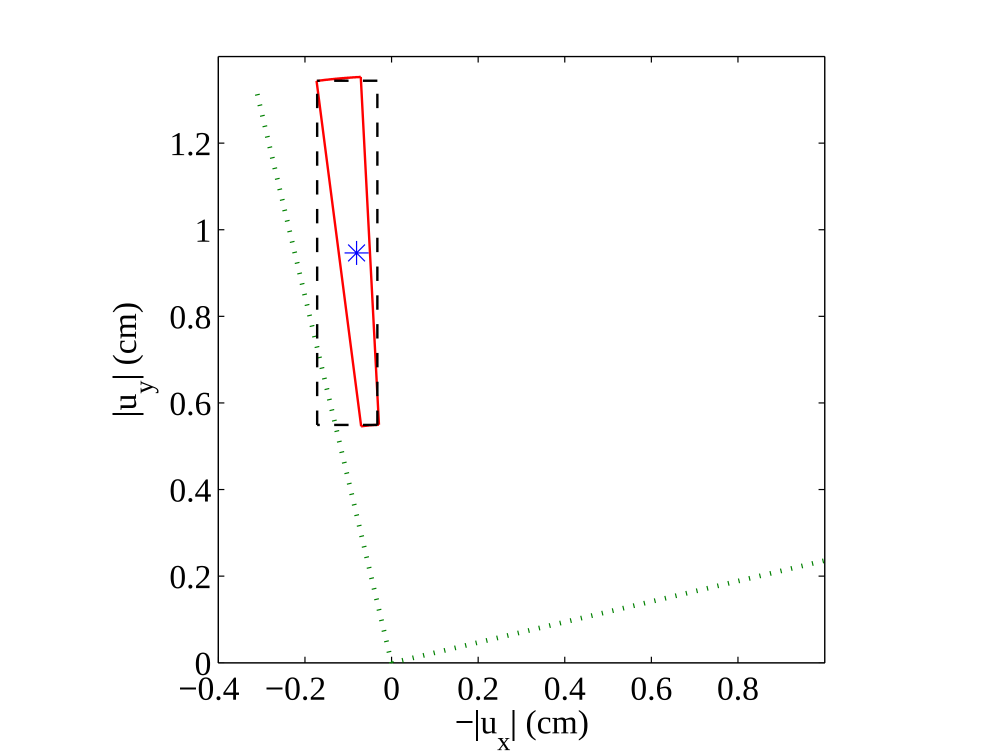
- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$ $(\omega = \omega_1^0)$

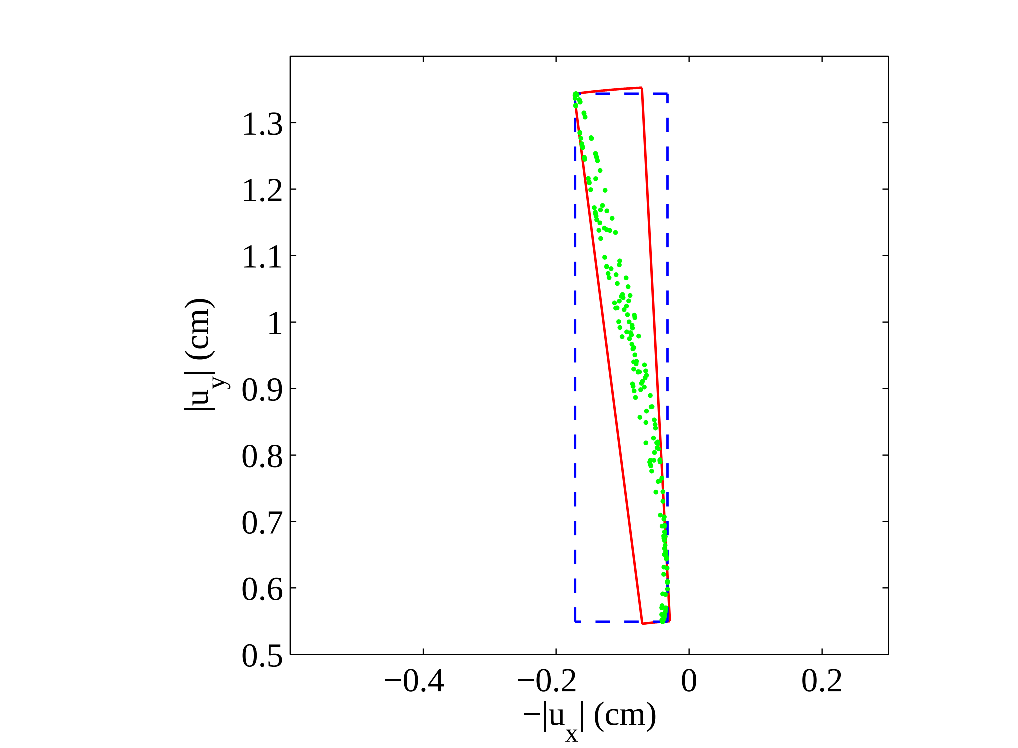




ex. (2DOF, 2-bar truss)

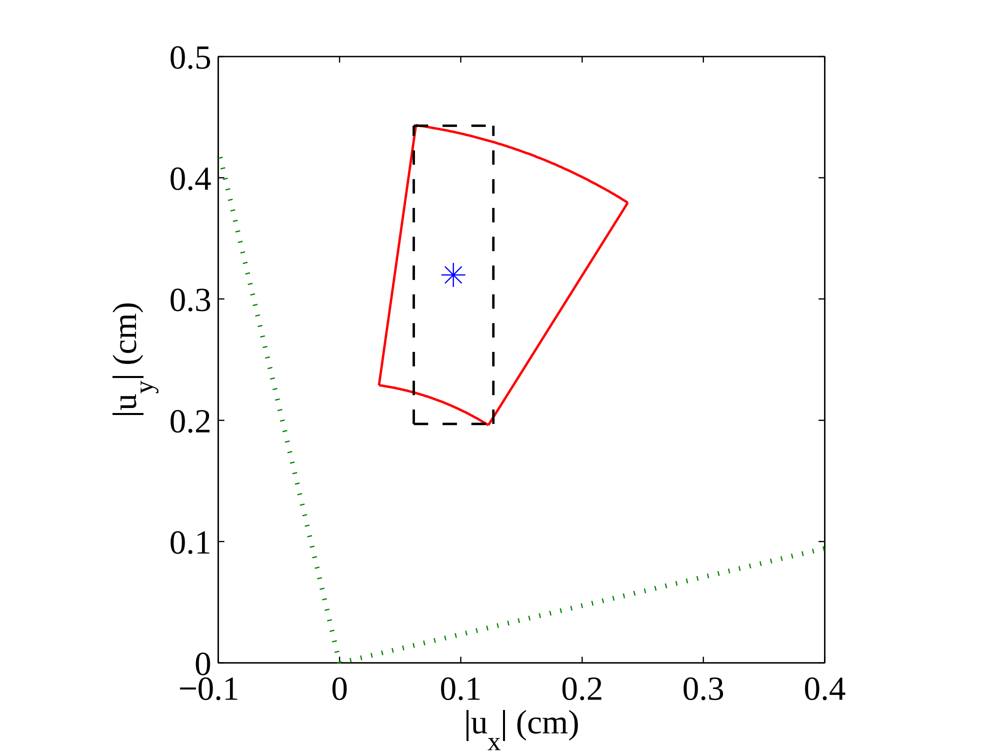
- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$ $(\omega = 1.1\omega_1^0)$

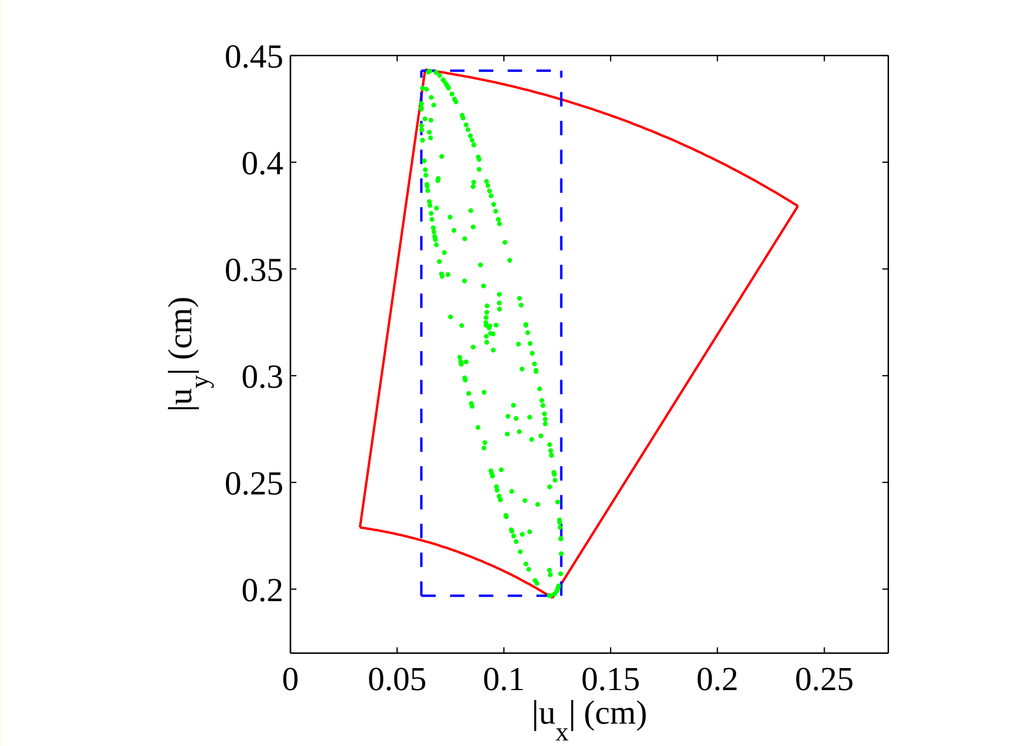




ex. (2DOF, 2-bar truss)

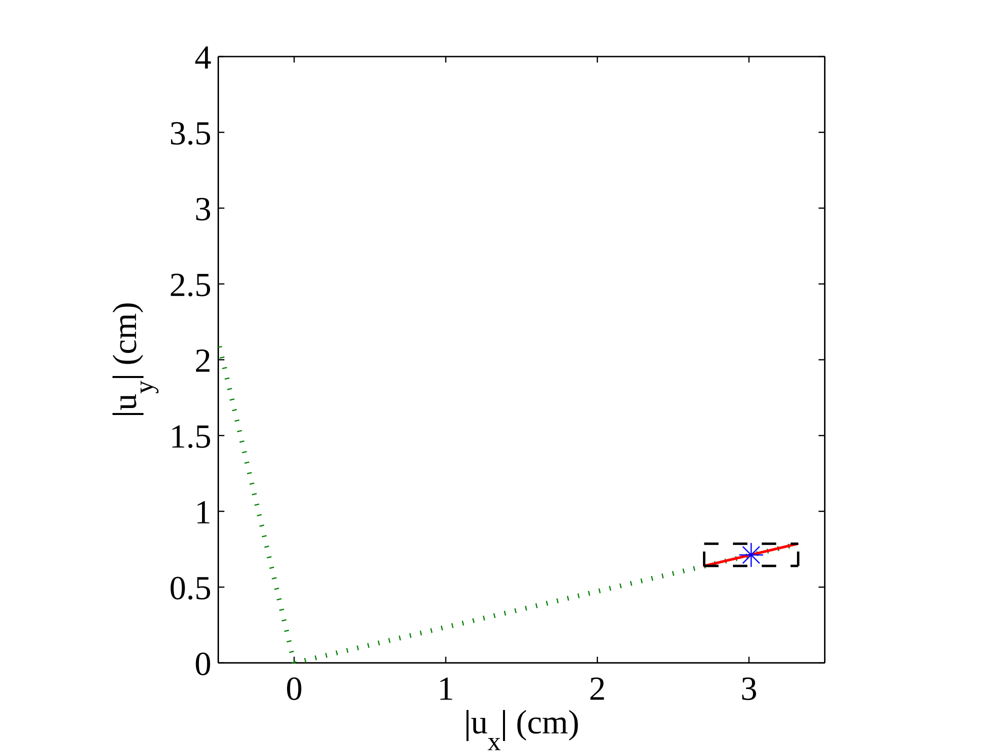
- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$ $(\omega = 1.3\omega_1^0)$

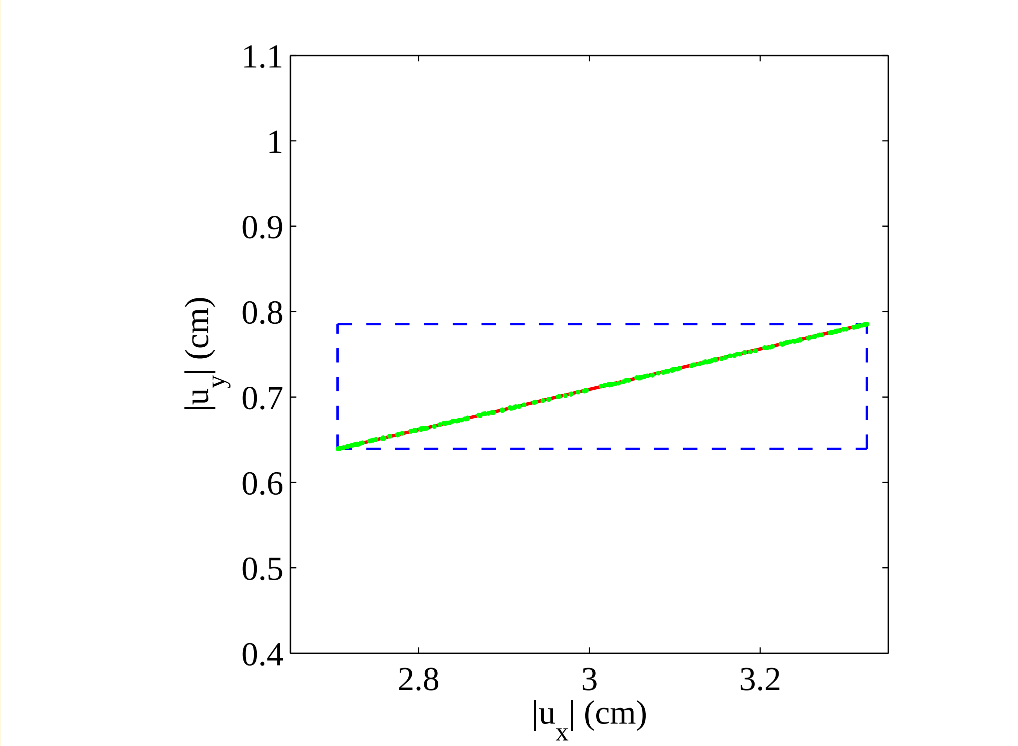




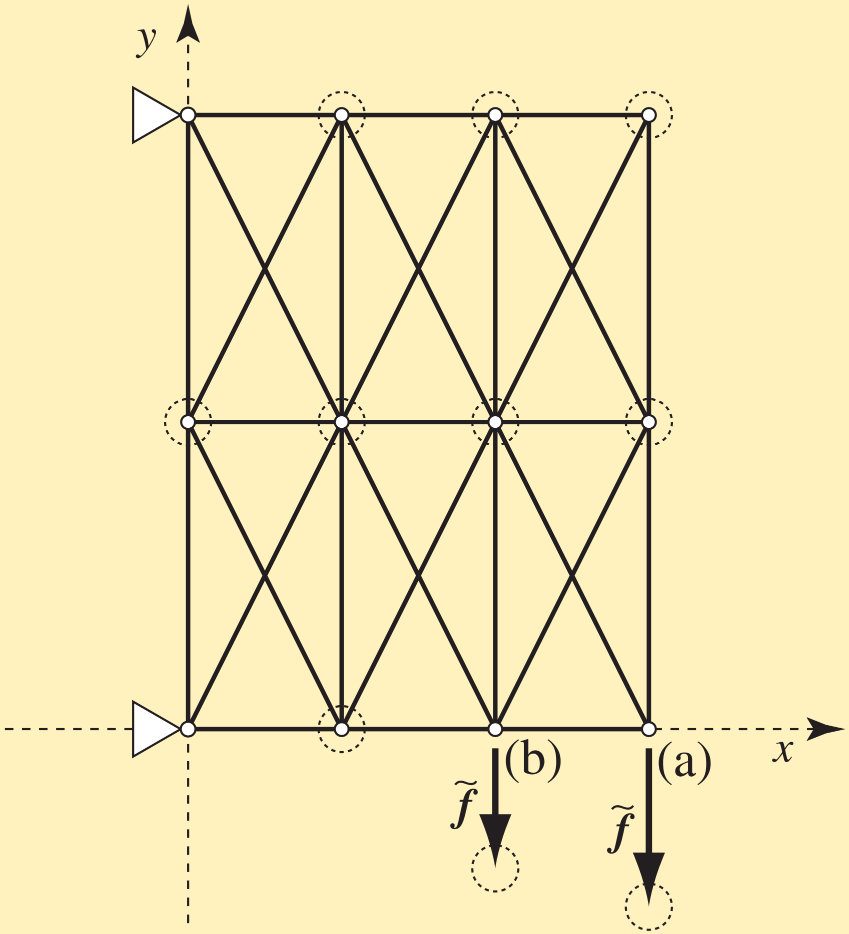
ex. (2DOF, 2-bar truss)

- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$ $(\omega = \omega_2^0)$





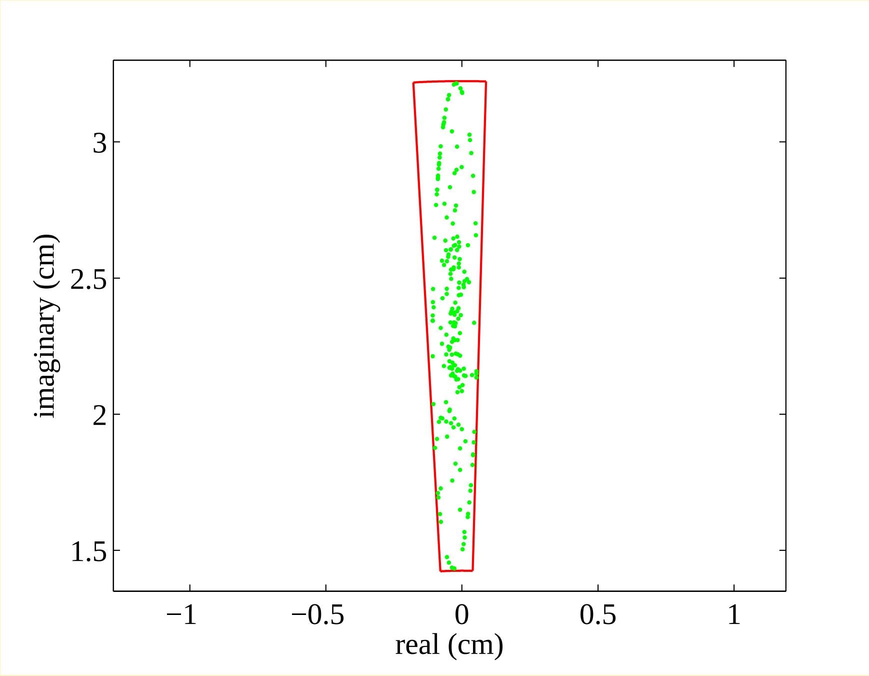
ex. (20DOF, 29-bar truss)



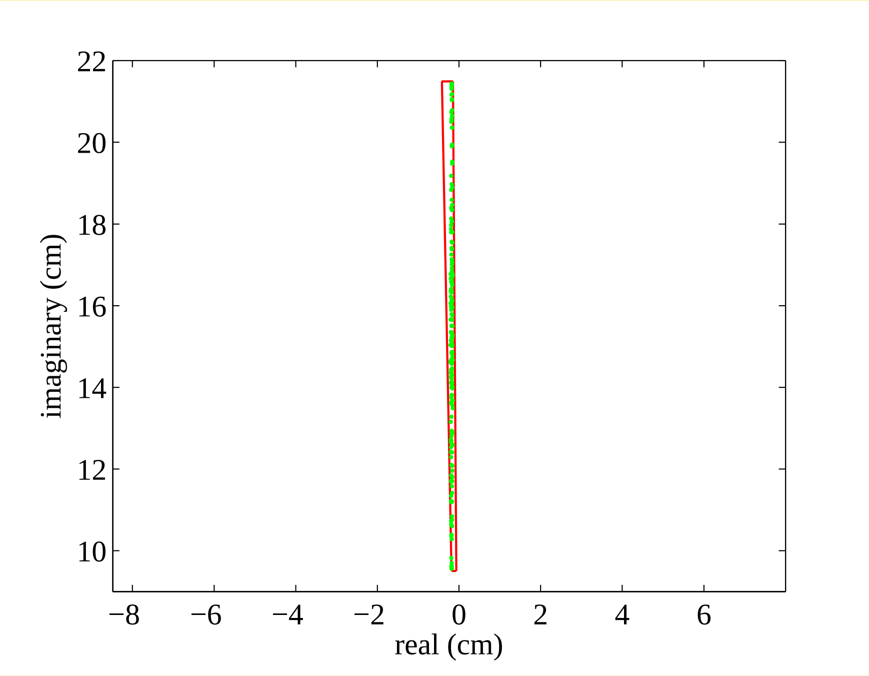
- nominal driving load
  $\tilde{f}$: 8 kN, 12 kN
- uncertainty nodal loads
  - at all the nodes
  - perturb independently
- complex damping:
  $\omega \boldsymbol{C} = 2\beta \boldsymbol{K}$ $(\beta = 0.02)$

ex. (20DOF, 29-bar truss)

- displacement amplitude (bottom-right node) $(\omega = \omega_1^0)$



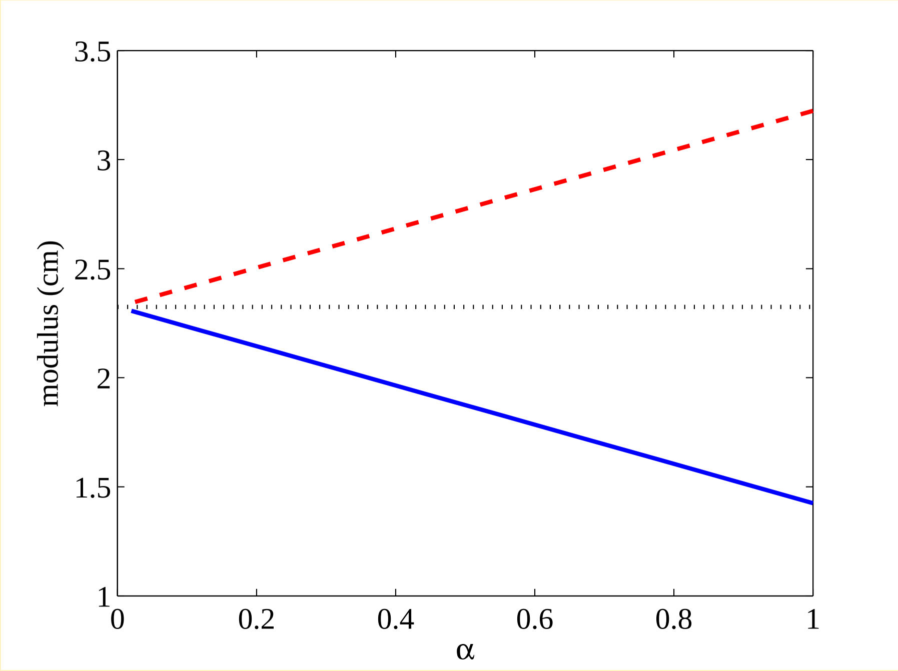
u_x



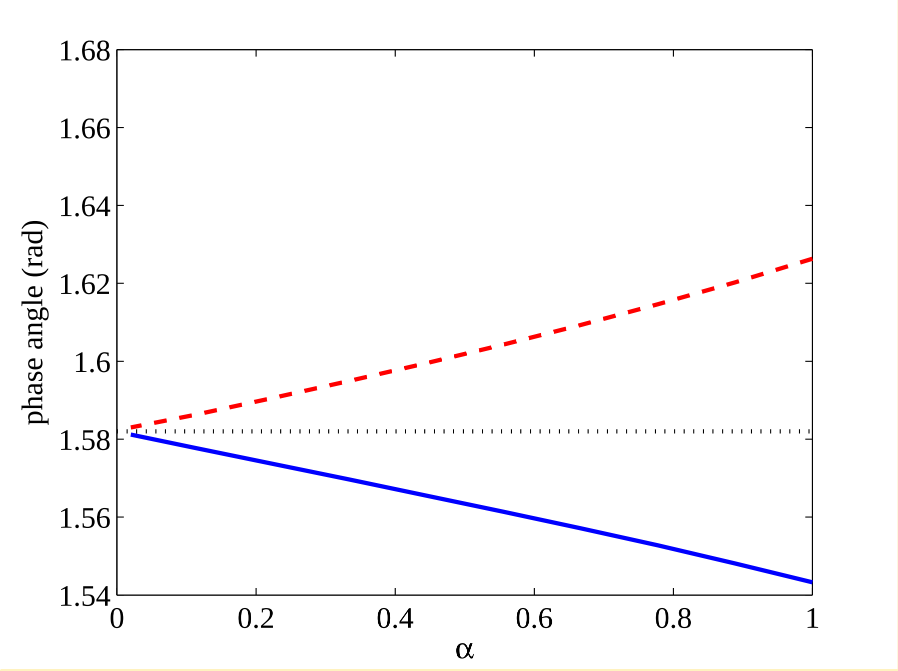
u_y

ex. (20DOF, 29-bar truss)

- displacement amplitude (bottom-right node)
  variations w.r.t. α ('magnitude' of uncertainty)



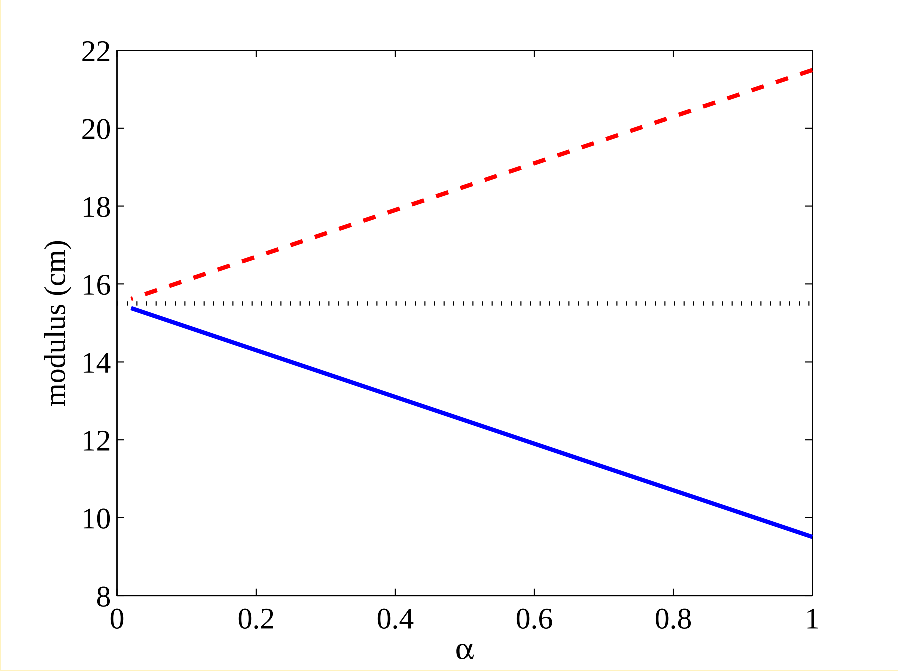
$|u_x|$



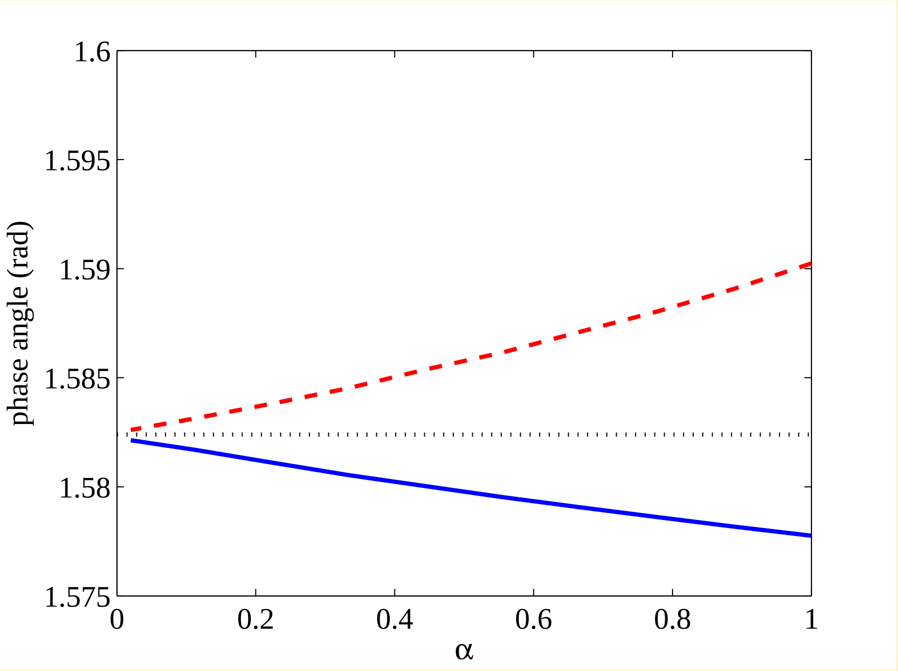
$\arg u_x$

ex. (20DOF, 29-bar truss)

- displacement amplitude (bottom-right node)
  variations w.r.t. α ('magnitude' of uncertainty)



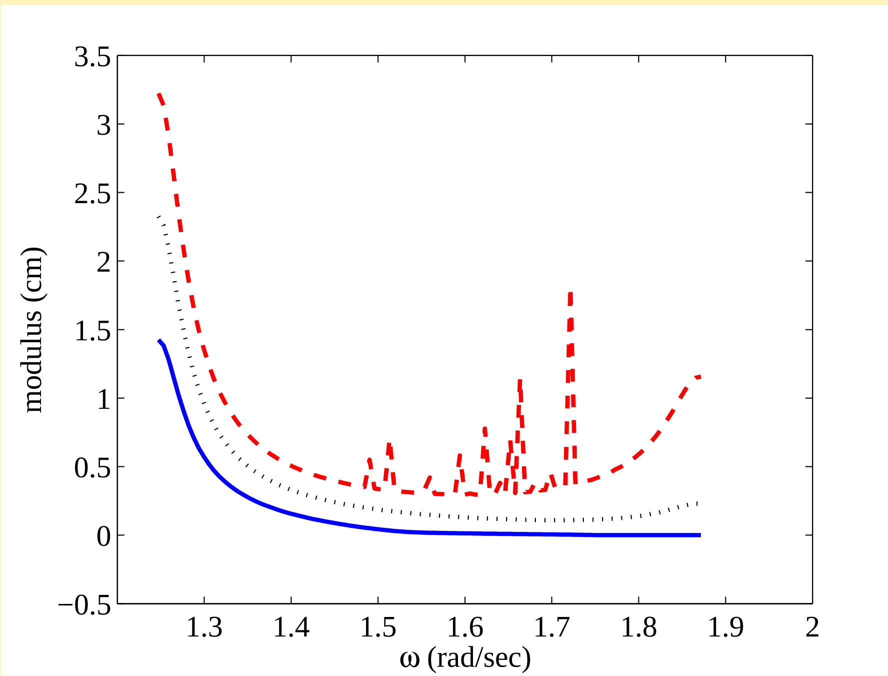
$|u_y|$



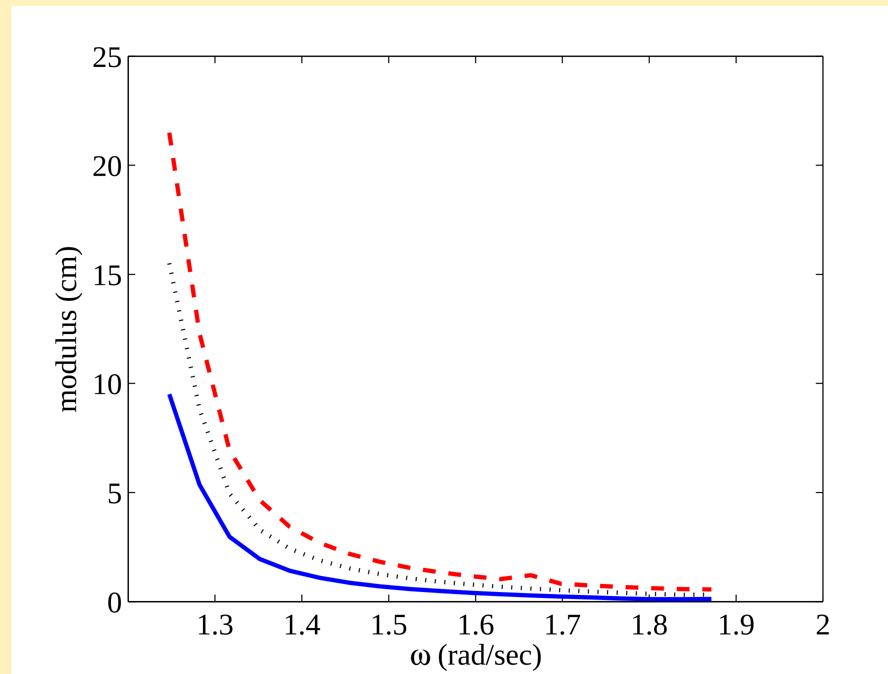
$\arg u_y$

ex. (20DOF, 29-bar truss)

- displacement amplitude (bottom-right node)
  variations w.r.t. ω (circular frequency)



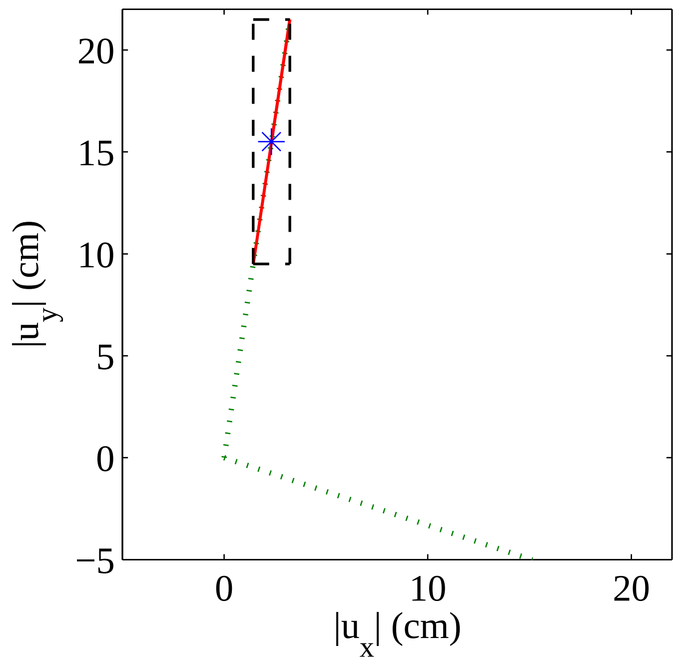
$|u_x|$

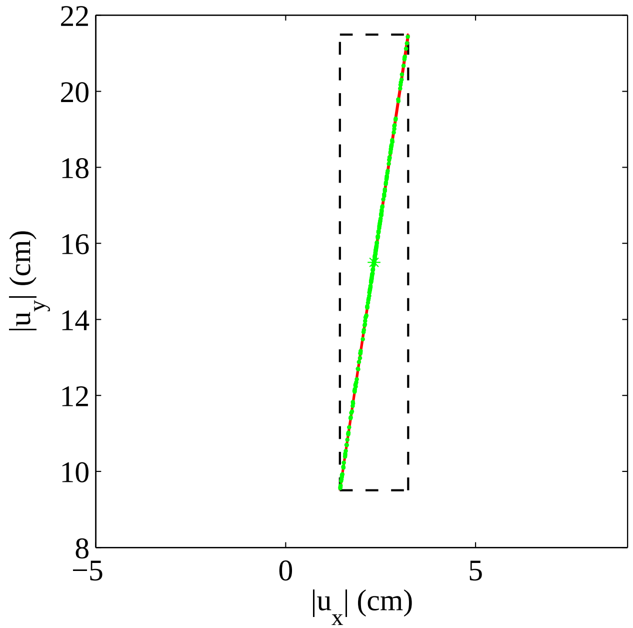


$|u_y|$

ex. (20DOF, 29-bar truss)

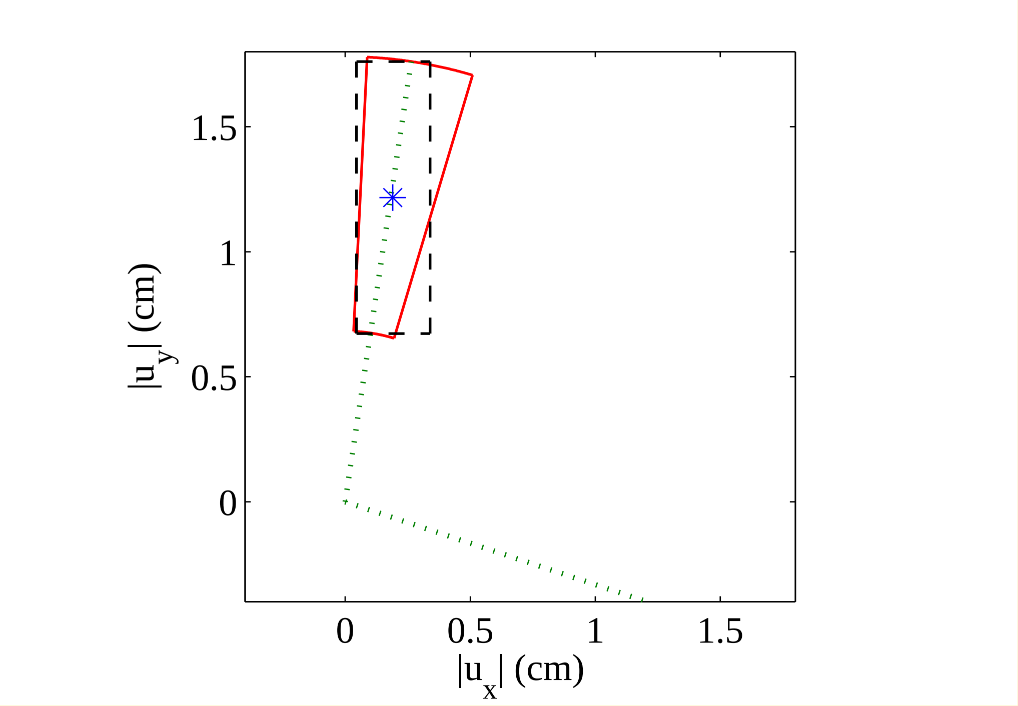
- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$ $\quad (\omega = \omega_1^0)$

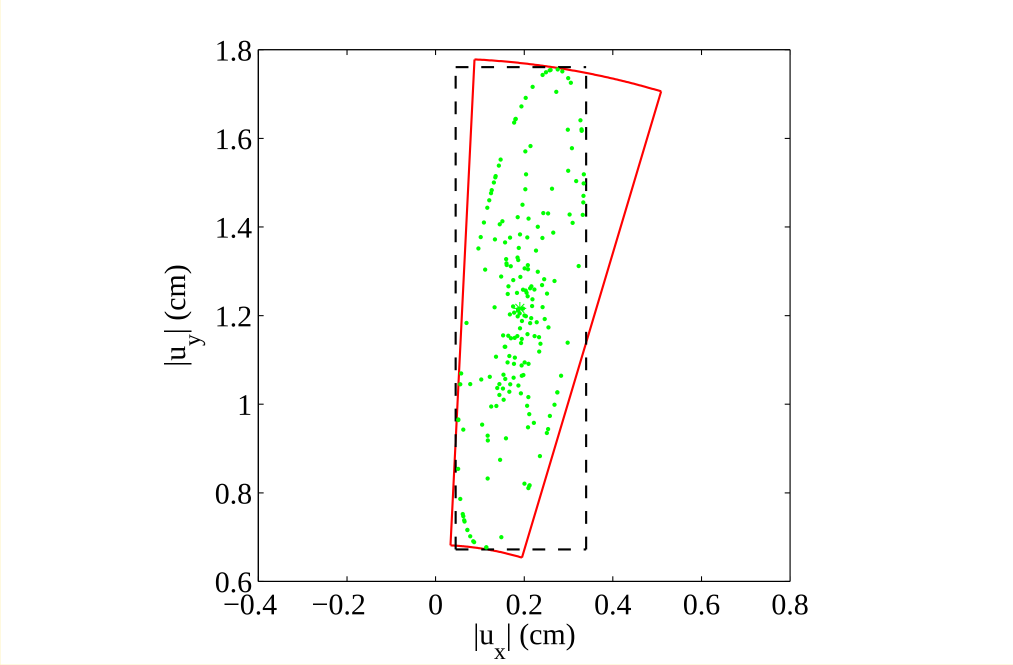




ex. (20DOF, 29-bar truss)

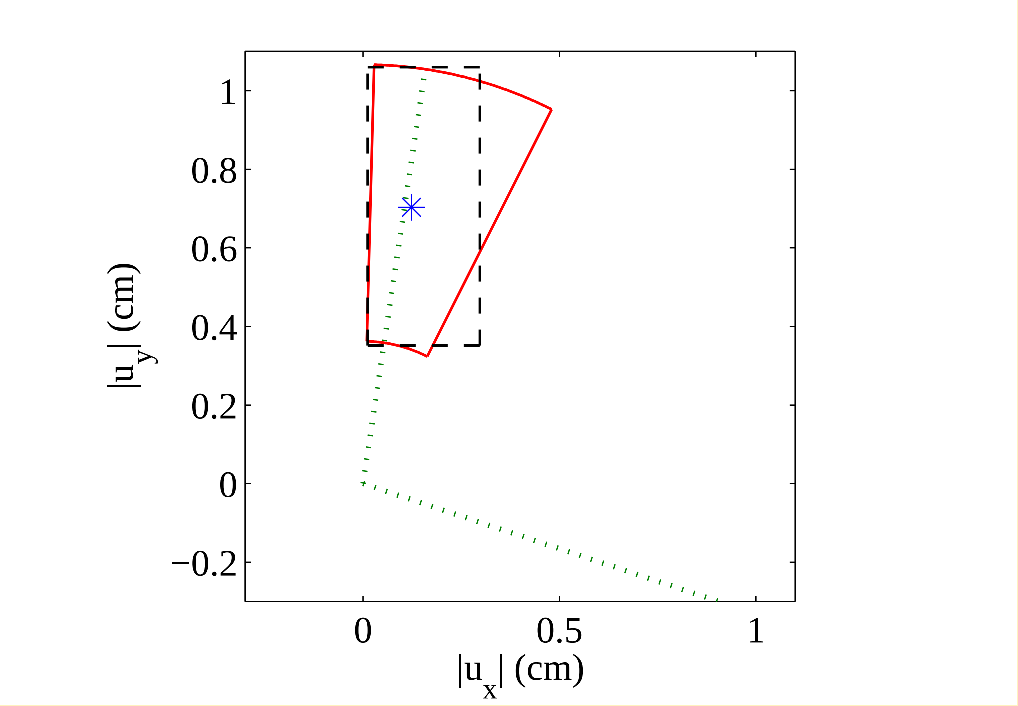
- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$ $(\omega = 1.2\omega_1^0)$

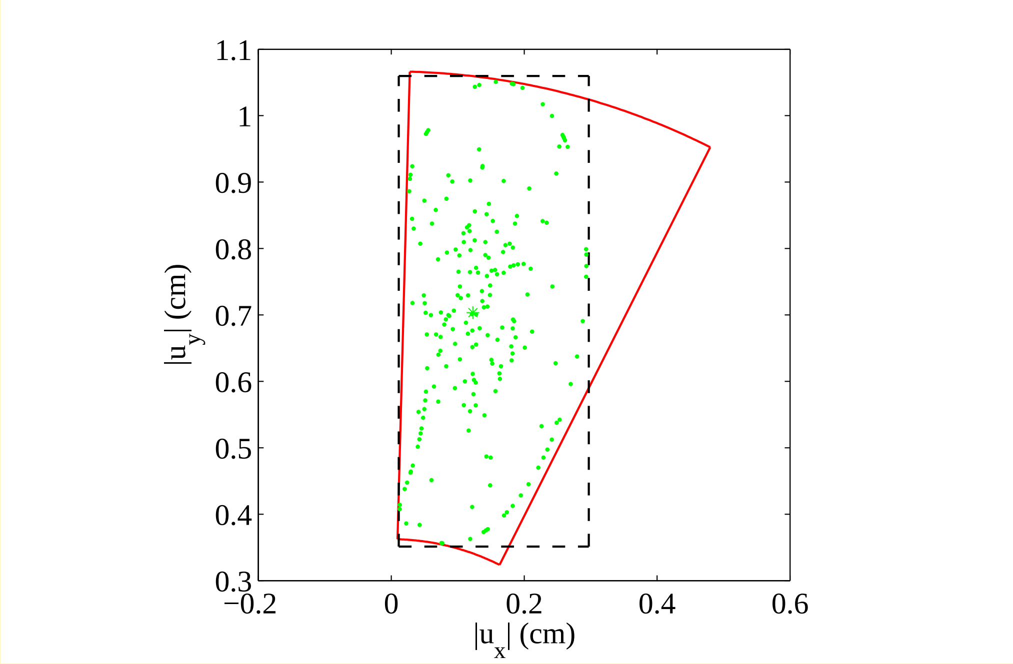




ex. (20DOF, 29-bar truss)

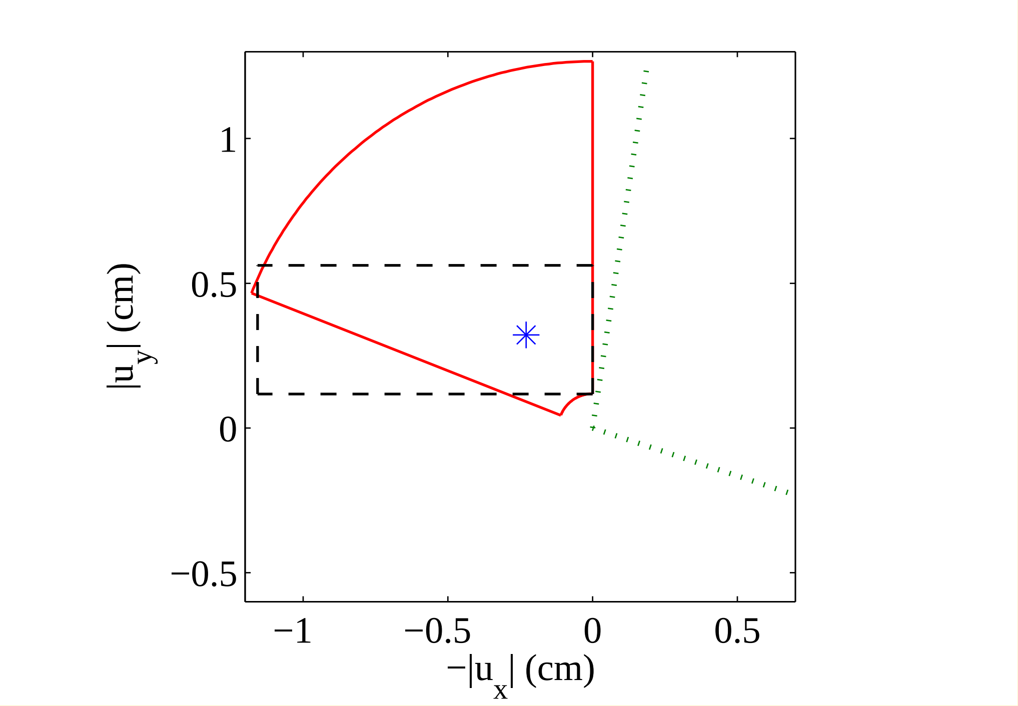
- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$ $(\omega = 1.3\omega_1^0)$

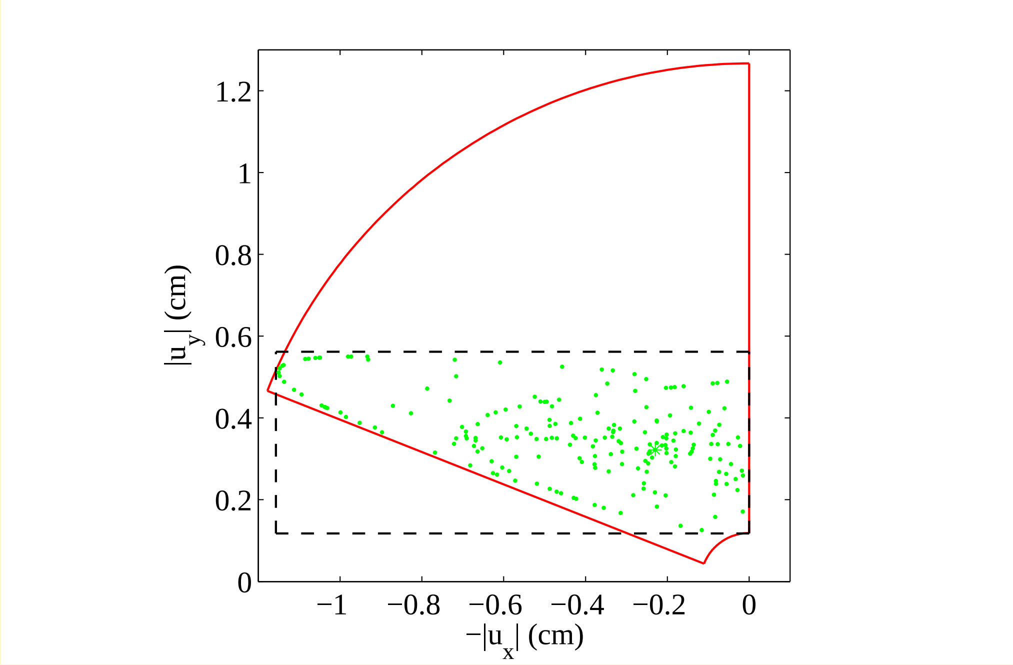




ex. (20DOF, 29-bar truss)

- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$ $(\omega = \omega_2^0)$

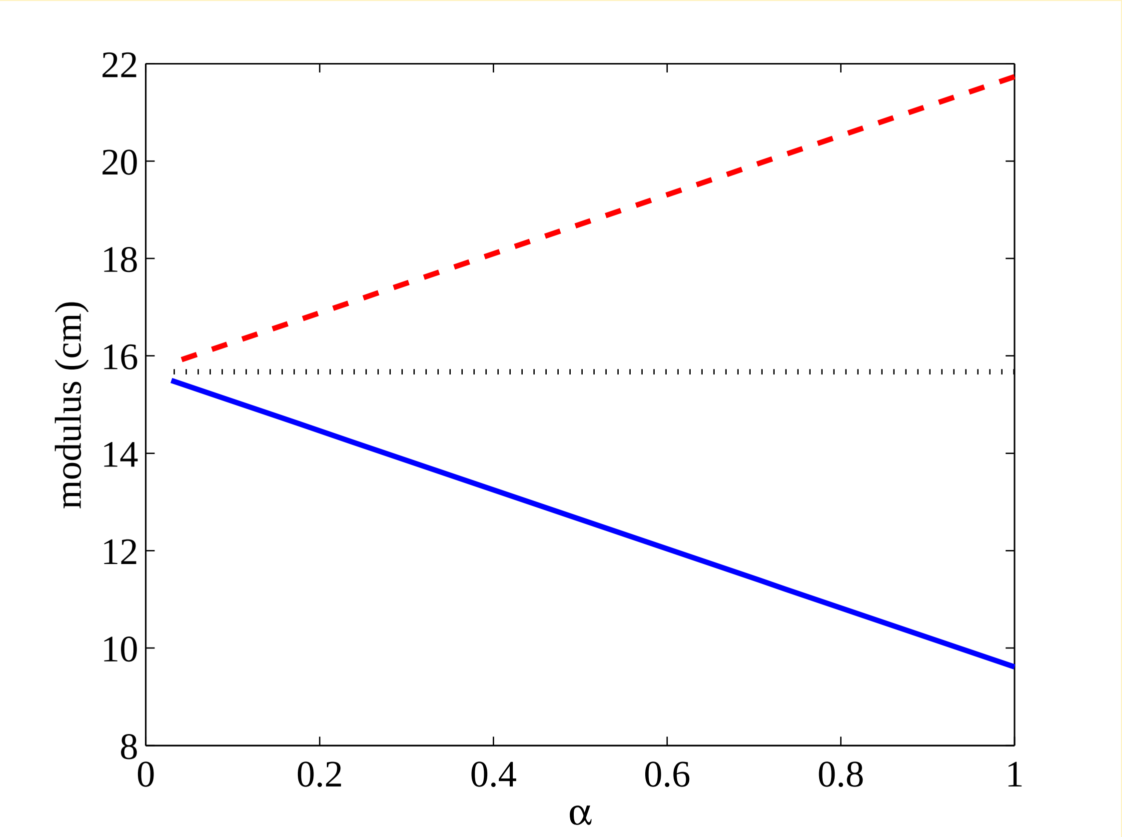




ex. (20DOF, 29-bar truss)

- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$

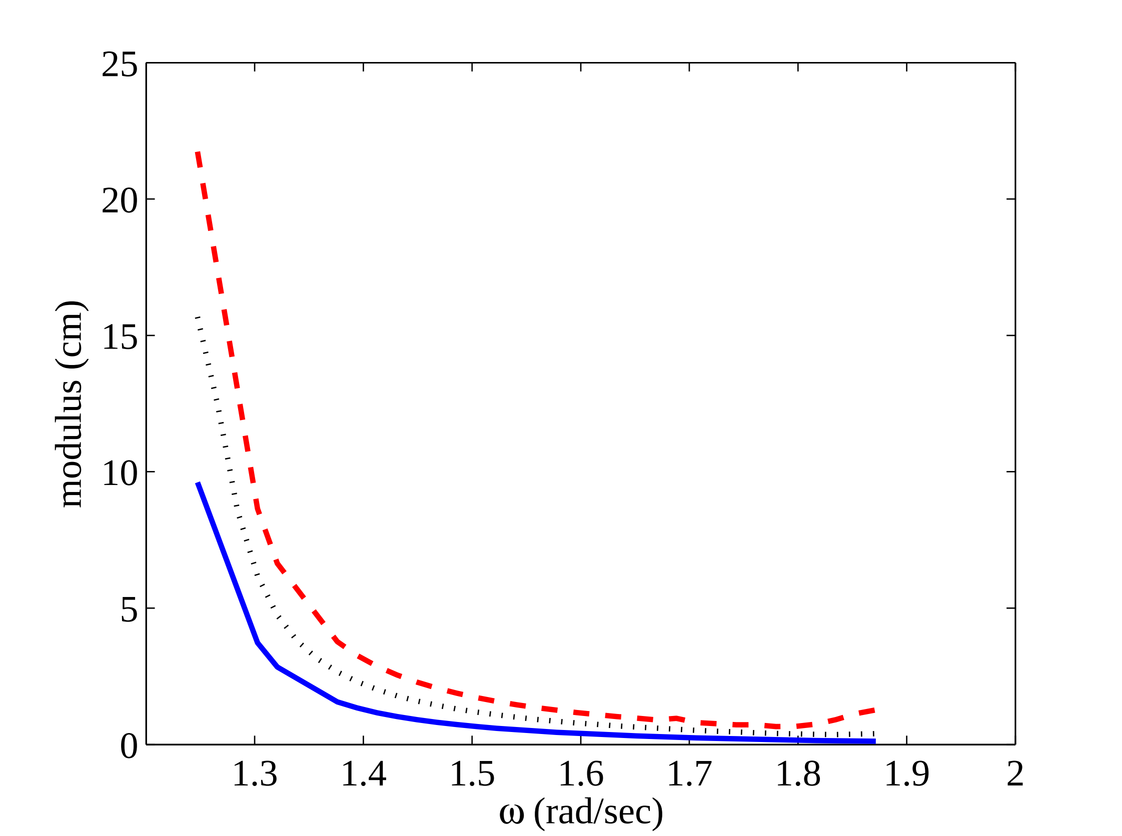
  variation w.r.t. α ('magnitude' of uncertainty)



ex. (20DOF, 29-bar truss)

- distribution of the vector $\begin{bmatrix} |u_x| \\ |u_y| \end{bmatrix}$

  variation w.r.t. ω (circular frequency)



conclusions

- uncertainty of dynamic load
  - uncertainty of harmonic exciting load
  - steady state of a damped structure
- bounds of response
  - modulus & argument of displacement amplitude
  - RO formulation
  - SDP approximation
  - conservativeness
    $\leftarrow$ an approximate optimal solution of (RO)